Balance Winds

- Geostrophic Balance
- Gradient Wind
- Cyclostrophic Wind Balance
- Balance with Friction / Ekman Balance

Geostrophic Balance

The equation of motion can be written as:

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} - \vec{F} + \vec{G}$$

where $\vec{\Omega}$ is the angular velocity of the earth (positive pointing upward from the north pole)

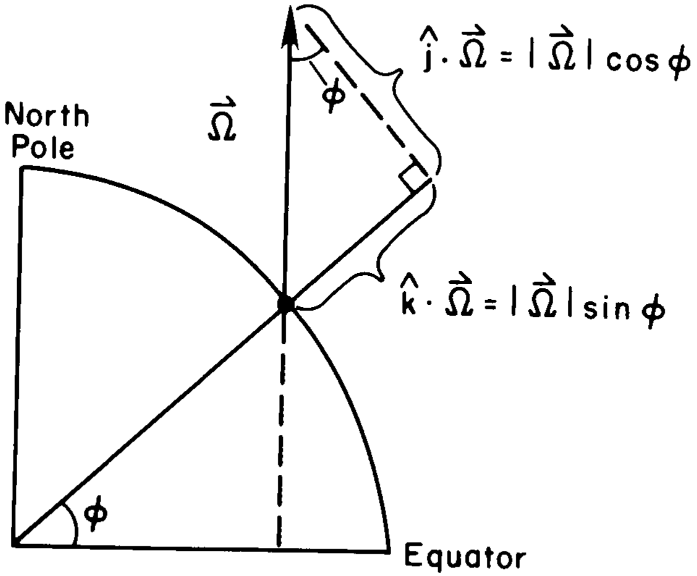
$|\Omega| = 2\pi/\text{day}$ corresponding to the rotation rate of the earth

$2\vec{\Omega}\times\vec{V}$ is called the Coriolis term

$\vec{F}$ represents the effects of friction

$\vec{G}$ is the gravitational vector ($\vec{G} = -g\vec{k}$)

Coriolis Term



$$-2\vec{\Omega}\times\vec{V} = -2\begin{vmatrix} i & j & k \\ 0 & |\Omega|\cos\phi & |\Omega|\sin\phi \\ u & v & w \end{vmatrix}$$

$$= -2\left[(\Omega\cos\phi w - \Omega\sin\phi v)\vec{i} + (\Omega\sin\phi u)\vec{j} + (-\Omega\cos\phi u)\vec{k}\right]$$

$$\therefore -2\vec{\Omega}\times\vec{V} = (fv - \hat{f}w)\vec{i} - fu\vec{j} + \hat{f}u\vec{k}$$

where $\hat{f} = 2\Omega\cos\phi$ and $f = 2\Omega\sin\phi$

The horizontal components of the equation of motion can therefore be written as :

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - \hat{f}w - Fu$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu - Fv$$

where $f = 2|\Omega|\sin\phi$ and $\hat{f} = 2|\Omega|\cos\phi$ and ϕ is the latitude

For the case of:

- no friction ($Fu = Fv \equiv 0$)
- no acceleration ($du/dt = dv/dt \equiv 0$)
- $|u|, |v| >> w$ which is typical on the synoptic scale

then

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad \Rightarrow \qquad \boxed{u = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \equiv u_g}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv \qquad \Rightarrow \qquad \boxed{v = \frac{1}{\rho f}\frac{\partial p}{\partial x} \equiv v_g}$$

where u_g and v_g are the geostrophic wind components *defined* by these relations. The geostrophic wind relation can be written in vector notation as:

$$\boxed{\vec{V}_g = \vec{k} \times \frac{1}{\rho f}\nabla_z p}$$

where ∇_z is the horizontal gradient operator $(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y})$

$$\vec{V}_g = \vec{k} \times \frac{1}{\rho f} \nabla_z p$$

This can be checked using the definition of the vector cross product :

$$\vec{V}_g = u_g \vec{i} + v_g \vec{j} = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ \frac{1}{\rho f} \frac{\partial p}{\partial x} & \frac{1}{\rho f} \frac{\partial p}{\partial y} & 0 \end{vmatrix}$$

$$\boxed{\vec{V}_g = \vec{k} \times \frac{1}{\rho f} \nabla_z p}$$

- The geostrophic wind is the balance between the pressure gradient force and the Coriolis force

- $\vec{V}_g$ must be horizontal and perpendicular to $\nabla_z p$ (the wind direction is parallel to the isobars)

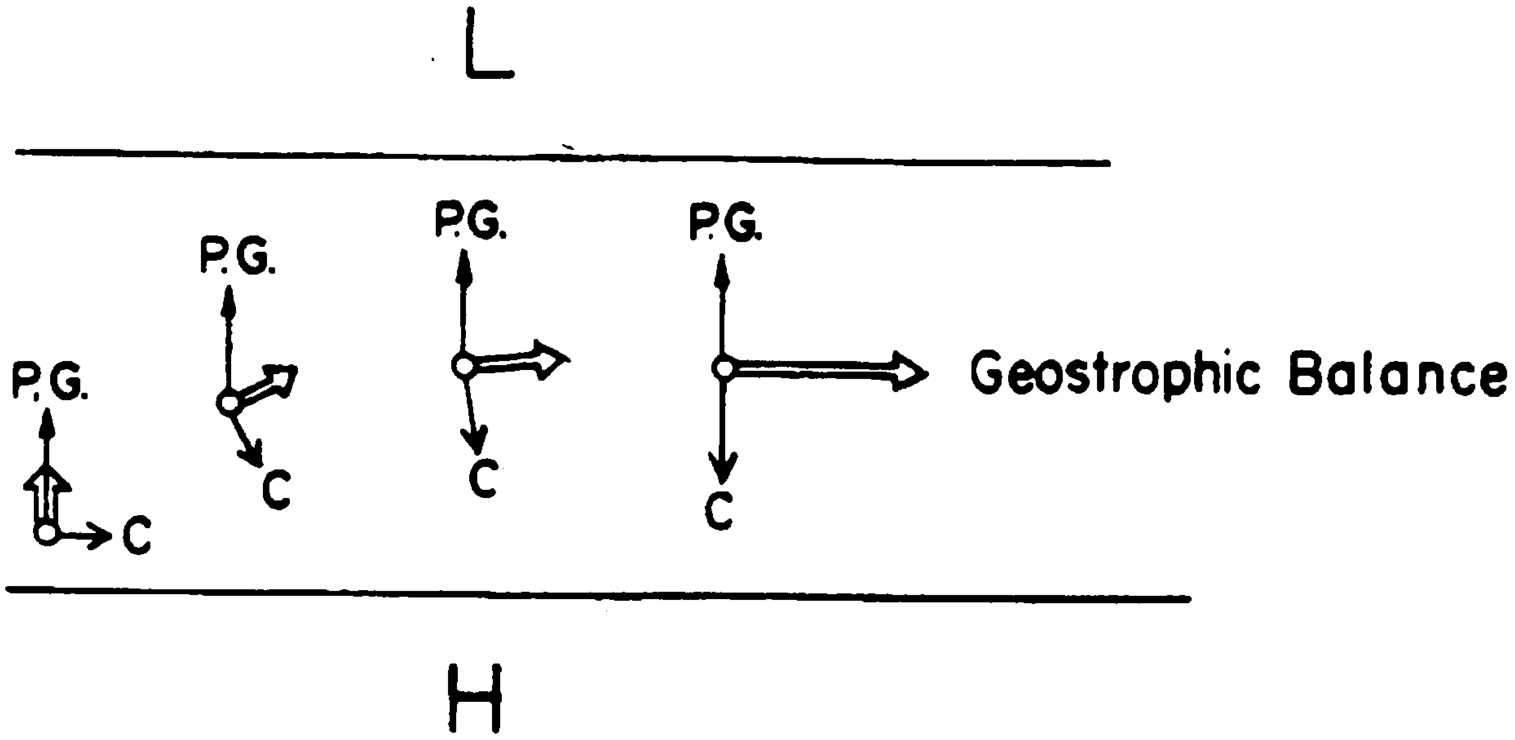
- $\vec{V}_g$ is directed such that the high pressure is to the right in the northern hemisphere and to the left in the southern hemisphere

- As ρ is nearly constant at a given height, $\vec{V}_g$ is almost linearly proportional to the pressure gradient (the larger $\nabla_z p$ the larger $\vec{V}_g$)

- A given value of $\nabla_z p$ will result in stronger $\vec{V}_g$ at lower latitudes because $f \rightarrow 0$ at low latitudes

- Geostrophic balance is rarely achieved at low latitudes

- It is assumed that the flow is straight for the geostrophic wind


L
P.G.
P.G.
P.G.
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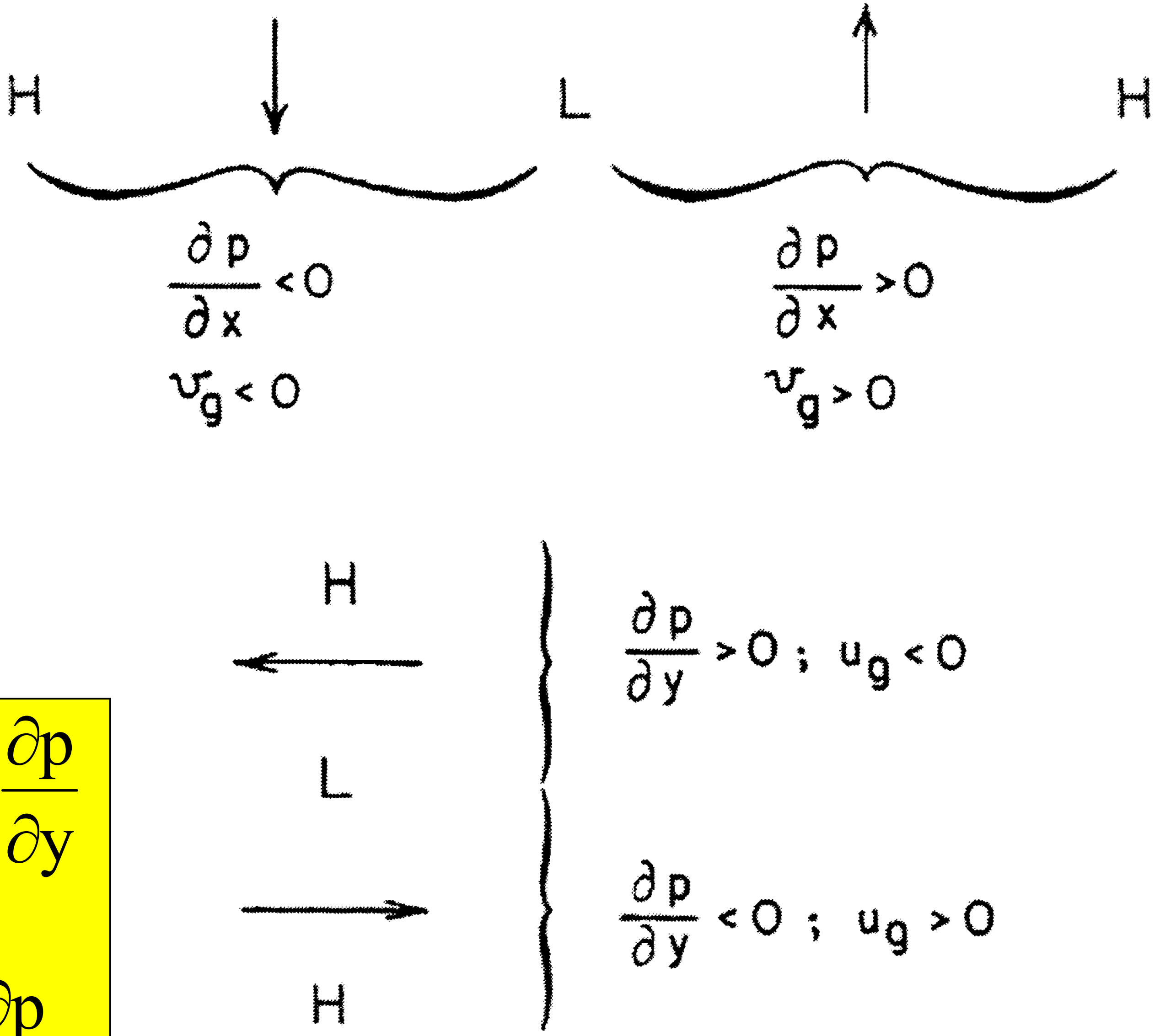
C
C
C
C
Geostrophic Balance
H



$$\frac{\partial p}{\partial x} < 0$$

$$v_g < 0$$

$$\frac{\partial p}{\partial x} > 0$$

$$v_g > 0$$

$$\frac{\partial p}{\partial y} > 0 \; ; \; u_g < 0$$

$$\frac{\partial p}{\partial y} < 0 \; ; \; u_g > 0$$

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y}$$

$$v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

Gradient Wind

- The geostrophic wind balance assumes that the wind flow is straight
- A more general form of a balanced wind can be obtained if accelerations due to curvature in the height or pressure fields are taken into account (remember that changes in the direction of the velocity vector with time result in acceleration)
- The gradient wind like the geostrophic wind is frictionless, but it is not unaccelerated

The equation of motion can be written as :

$$\frac{\mathrm{d}\vec{\mathrm{V}}}{\mathrm{dt}} = -\frac{1}{\rho}\nabla p - 2\vec{\Omega}\times\vec{V} - \vec{F} + \vec{G}$$

As we now have acceleration due to the effects of curvature,

the term $\frac{\mathrm{d}\vec{\mathrm{V}}}{\mathrm{dt}}$ is not zero as it was for the geostrophic wind.

Following Pielke's notes we can obtain an expression for the for the acceleration due to curvature :

$$\boxed{\frac{\mathrm{d}\vec{\mathrm{V}}}{\mathrm{dt}} = \frac{\left|\vec{V}_{gr}\right|^2}{R_T}(-\vec{r})}$$

where the subscript gr stands for gradient wind

$\vec{\mathrm{r}}$ is the position vector

R_{T} is the magnitude of $\vec{\mathrm{r}}$

The right hand side of the above equation is simply the centrifugal force

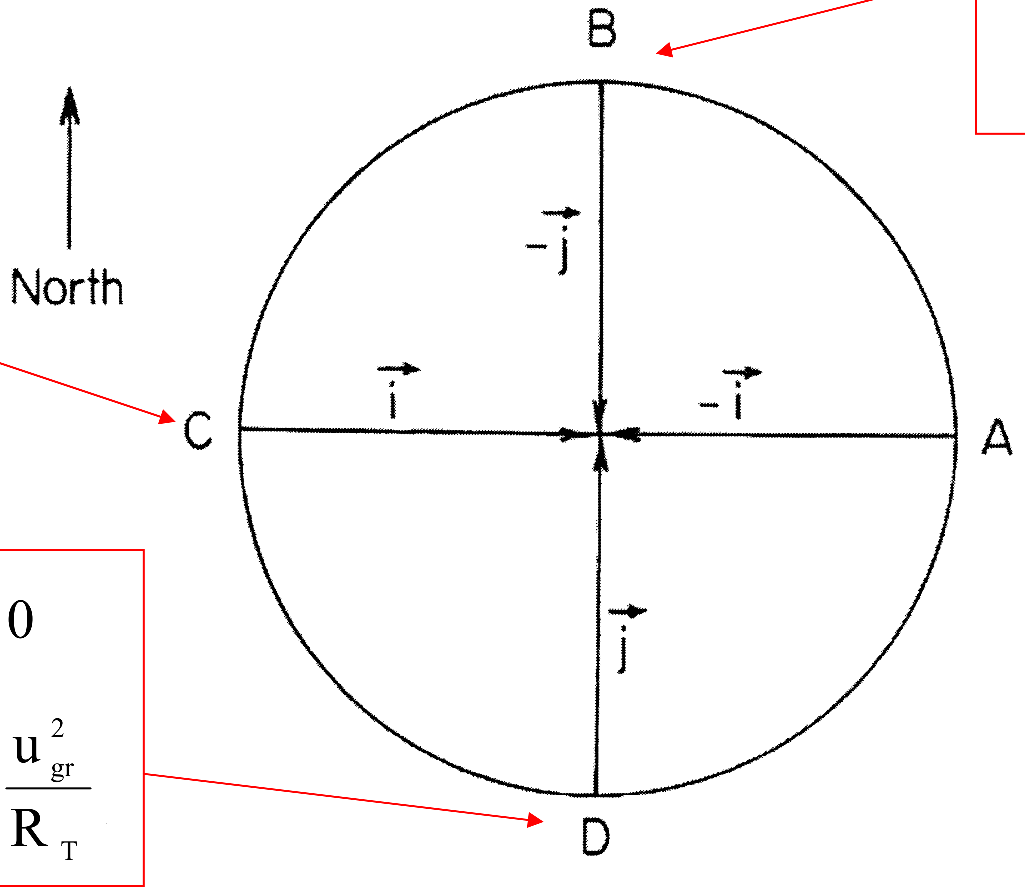
$$\frac{d\vec{V}}{dt} = \frac{\left|\vec{V}_{gr}\right|^2}{R_T}(-\vec{r}) \Rightarrow \frac{du}{dt}\vec{i} + \frac{dv}{dt}\vec{j} = \frac{u_{gr}^2 + v_{gr}^2}{R_T}(-\vec{r})$$

$$\frac{du}{dt} = 0$$

$$\frac{dv}{dt} = \frac{-u_{gr}^2}{R_T}$$

$$\frac{dv}{dt} = 0$$

$$\frac{du}{dt} = \frac{v_{gr}^2}{R_T}$$



$$\frac{dv}{dt} = 0$$

$$\frac{du}{dt} = \frac{-v_{gr}^2}{R_T}$$

$$\frac{du}{dt} = 0$$

$$\frac{dv}{dt} = \frac{u_{gr}^2}{R_T}$$

The direction of the unit vector $-\vec{r}$ at points A, B, C and D are denoted by the appropriate Cartesian unit vector

- To investigate the balanced wind which develops when the acceleration due to curvature is included along with the Coriolis force and the pressure gradient force, we will focus on point A. There is no loss of generality as the coordinate system can always be rotated so that a point of interest corresponds to location A.

- At point A, $u_g = 0$, while $v_g > 0$ for a low and $v_g < 0$ for a high pressure in the northern hemisphere. Substituting into:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv$$

and ignoring friction and the fw Coriolis term we get:

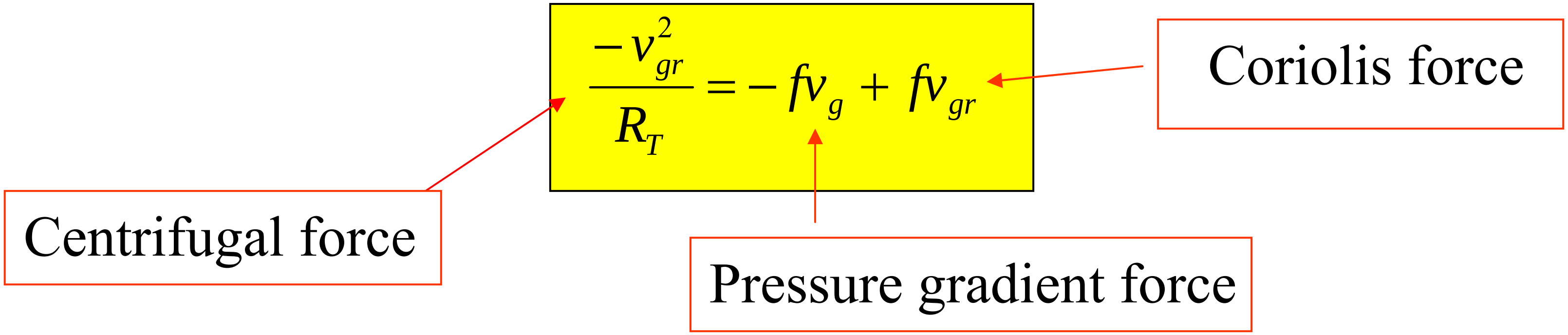
$$\frac{-\mathrm{v}_{\mathrm{gr}}^{2}}{\mathrm{R}_{\mathrm{T}}} = -\frac{1}{\rho}\frac{\partial \mathrm{p}}{\partial \mathrm{x}} + \mathrm{f}\mathrm{v}_{\mathrm{gr}}$$

$$\frac{-v_{gr}^2}{R_T} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv_{gr}$$

$$Now\ \text{as}\ \frac{\text{-1}}{\rho}\frac{\partial p}{\partial x} = -fv_g\ \text{we get:}$$

$$\boxed{\frac{-v_{gr}^2}{R_T} = -fv_g + fv_{gr} = f(v_{gr} - v_g)}$$

- The velocity v$_{gr}$ which solves this relation is called the GRADIENT WIND



$$\boxed{\frac{-v_{gr}^2}{R_T} = -fv_g + fv_{gr}}$$

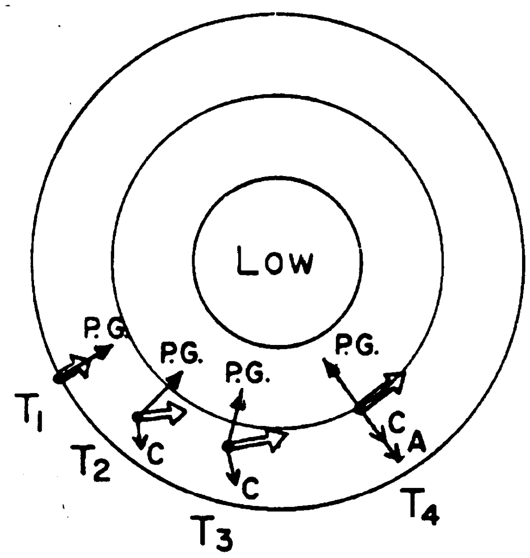
Coriolis force

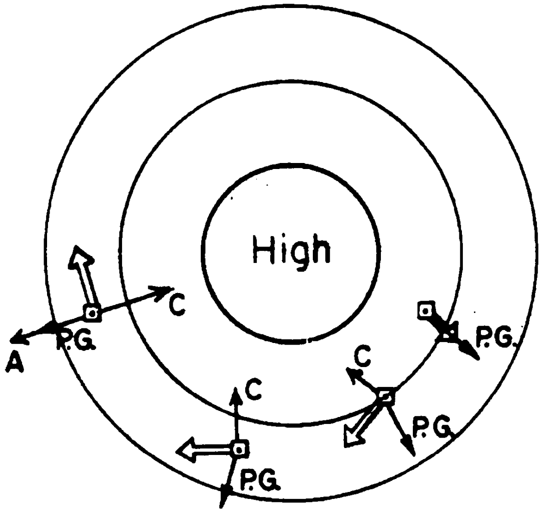
Centrifugal force

Pressure gradient force

- The gradient wind balance is a three-way balance between the Coriolis force, the centrifugal force and the horizontal pressure gradient force

Gradient Balance





$$\frac{-v_{gr}^2}{R_T} = -fv_g + fv_{gr} = f(v_{gr} - v_g)$$

Rearranging this equation :

$$\mathrm{v}_{\mathrm{gr}}^2 + fR_T v_{gr} - R_T fv_g = 0$$

Using the quadratic equation formula to solve for v_{gr} :

$$\boxed{v_{gr} = (-fR_T \pm \sqrt{f^2 R_T^2 + 4R_T fv_g})/2}$$

- For a cyclone in the northern hemisphere, $\mathrm{v}_\mathrm{g} > 0$ at A so that the radical is always real => no limit to the magnitude of the gradient wind

- For an anticyclone in the northern hemisphere, $\mathrm{v}_\mathrm{g} < 0$ at A so that $\mathrm{f}^2\mathrm{R}_\mathrm{T}^2 > 4\mathrm{R}_\mathrm{T}\mathrm{fv}_\mathrm{g}$ => $\mathrm{v}_\mathrm{g} < \mathrm{fR}_\mathrm{T}/4$ for the radical to be real => there is a constraint on the magnitude of the pressure gradient force in anticyclones that does not exist for low pressures. This is the reason that lows on synoptic weather maps often have tight gradients while highs don't.

$$v_{gr}^2 + fR_T v_{gr} - R_T f v_g = 0 \text{ can be rewritten as :}$$

$$\boxed{\frac{v_{gr}^2}{R_T f} + v_{gr} = v_g}$$

- $v_{gr} < v_g$ for a cyclone as $v_g > 0$ at A

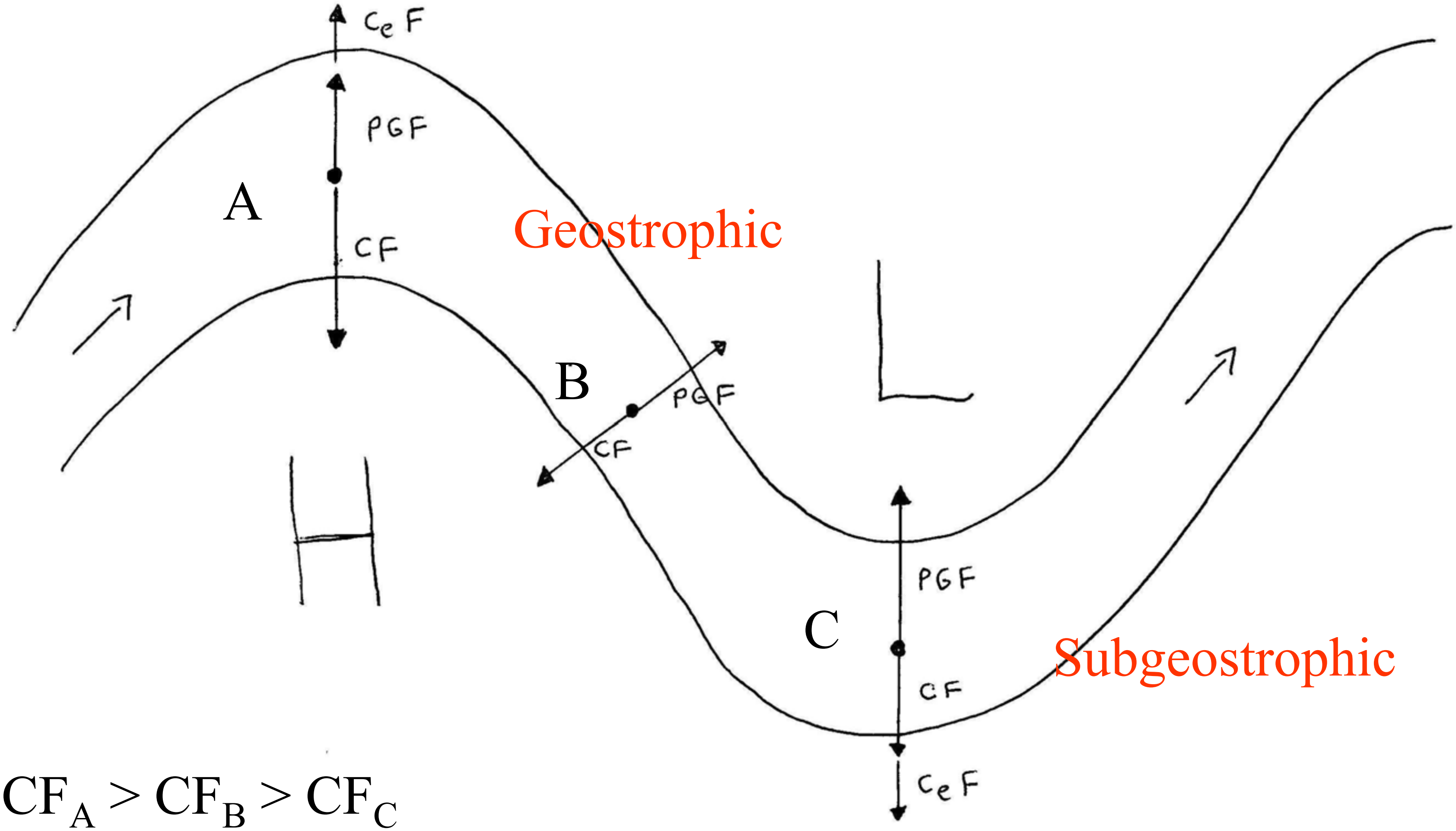
- $|v_{gr}| > |v_g|$ for an anticyclone as $v_g < 0$ at A

- These inequalities show that for the same pressure gradient (as represented by the geostrophic wind), the gradient balanced wind is stronger around a high than a low

- The gradient winds associated with a cyclone are SUBGEOSTROPHIC because the centrifugal force helps to balance the acceleration due to the pressure gradient force, therefore the Coriolis terms fv_{gr} and v_{gr} don't need to be as large

- The winds associated with an anticyclone are SUPERGEOSTROPHIC because a large Coriolis acceleration (and hence large value of v_{gr}) is needed to balance the sum of the acceleration due to the pressure gradient force and the centrifugal force

Supergeostrophic



A

Geostrophic

B

C

Subgeostrophic

$CF_A > CF_B > CF_C$

$V_{grA} > V_{grB} > V_{grC}$

Cyclostrophic Wind Balance

- When Coriolis force is neglected in the gradient wind balance we obtain a balance between the centrifugal force and the pressure gradient force – this balance is called the cyclostrophic wind balance
- Used to estimate wind speeds in small-scale vortices such as tornadoes and dust devils

We had $\frac{-v_{gr}^2}{R_T} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv_{gr}$ for the gradient wind

For cyclostrophic wind balance:

$$\boxed{\frac{-v_d^2}{R_T} = -\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

The cyclostrophic wind can be obtained from:

$$\mathrm{v_d} = \pm\sqrt{\frac{R_T}{\rho}\frac{\partial p}{\partial x}}$$

A tornado with a radius of 0.5 km, a pressure gradient of 100 mb km^{-1} and an air density of 1.25 kg m^{-3} would have a cyclostrophic wind of 63 m s^{-1}

Balance with Friction

- To evaluate the impact of friction on the resultant wind balance, we can retain the friction terms in our horizontal equations of motion

Setting

$\mathrm{F_u} = C_D u^2$ and $\mathrm{F_v} = C_D v^2$

we obtain a more general form of the gradient wind balance:

$$\boxed{\frac{\mathrm{V_F^2}}{\mathrm{R_T}} + fV_F = fV_g + C_D V_F^2}$$

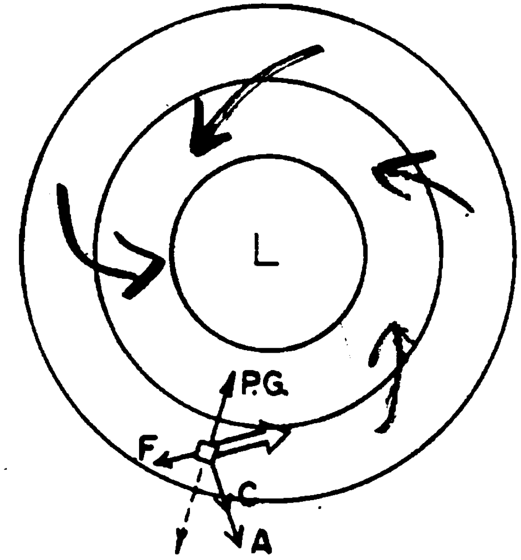
where $\mathrm{V_F}$ is in general at some angle to the gradient wind and the subscript F indicates that the effects of friction have been included. $\mathrm{C_D}$ is a drag coefficient which is a function of height above the ground and the thermodynamic stability

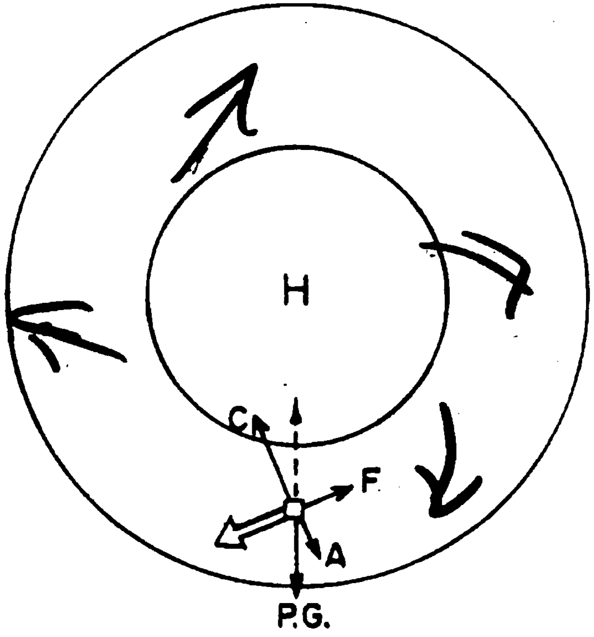
- Friction decelerates the flow and this turns the wind towards low pressure => results in low-level divergence out of anticyclones and low-level convergence into cyclones

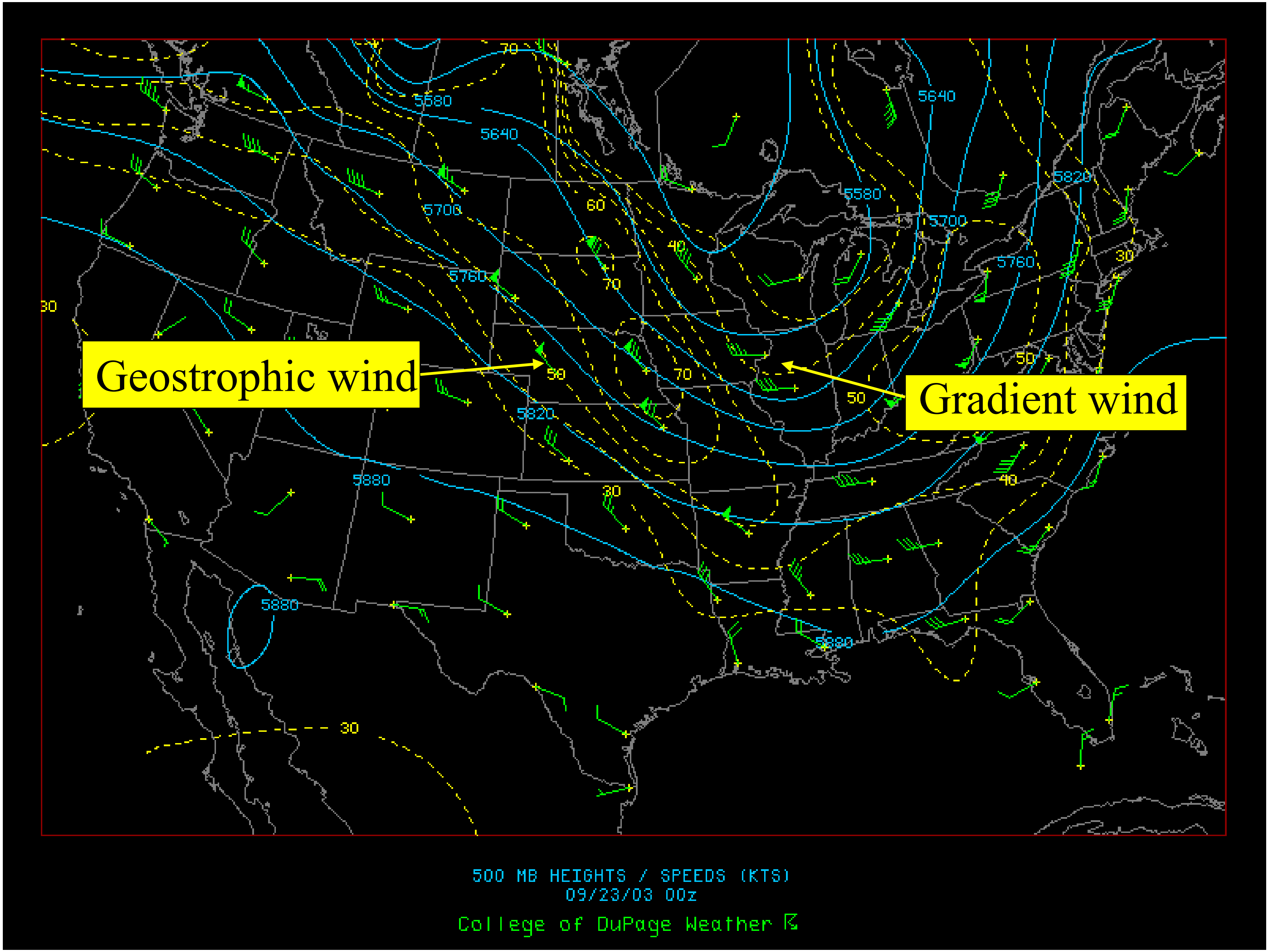
- The frictional acceleration acts directly opposite to the direction of the wind

- The Coriolis acceleration is perpendicular to the wind direction

- The centrifugal force is also perpendicular to the instantaneous wind direction





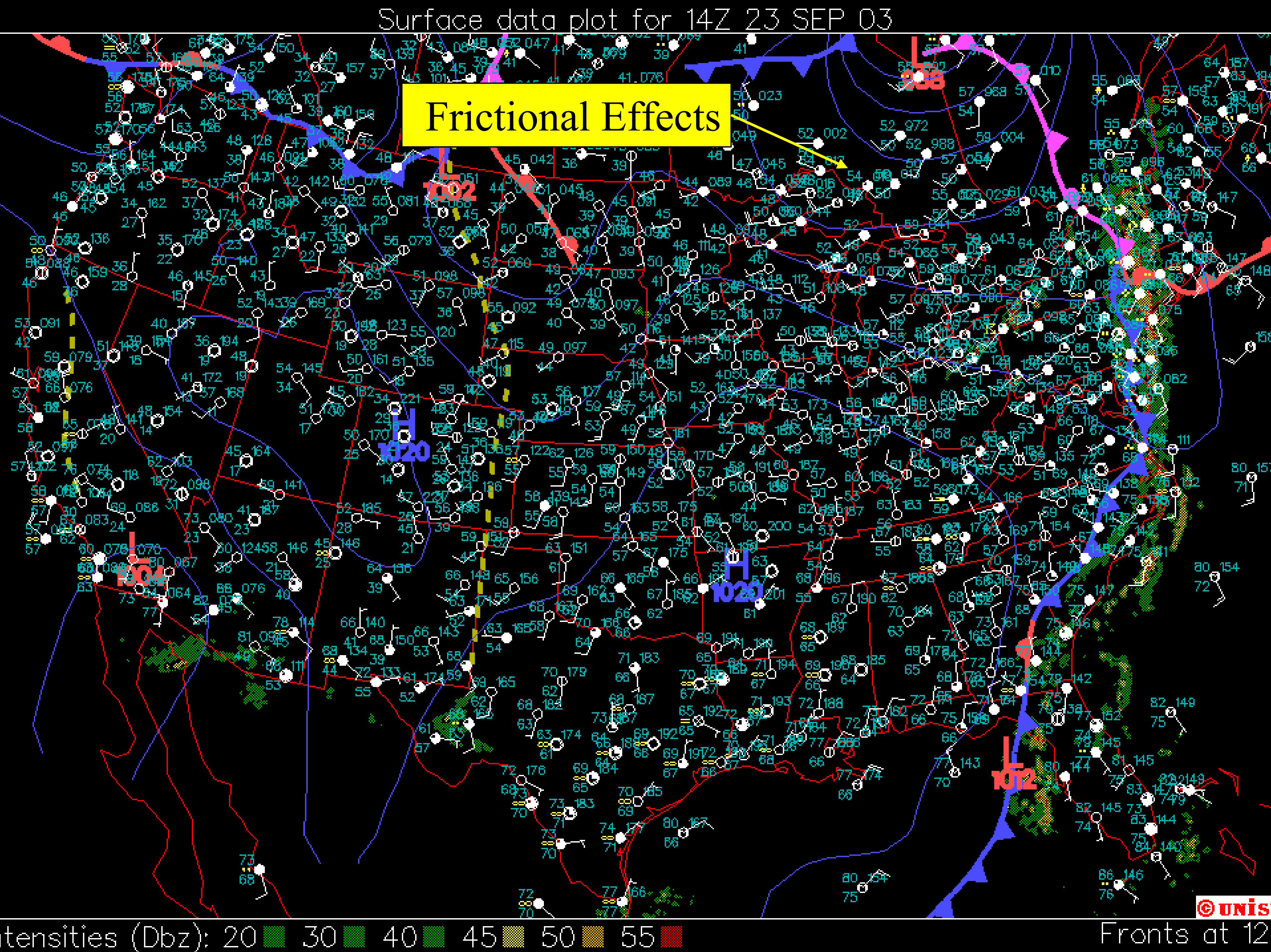


Geostrophic wind

Gradient wind

500 MB HEIGHTS / SPEEDS (KTS)
09/23/03 00z

College of DuPage Weather



Frictional Effects

Summary of Balance Winds

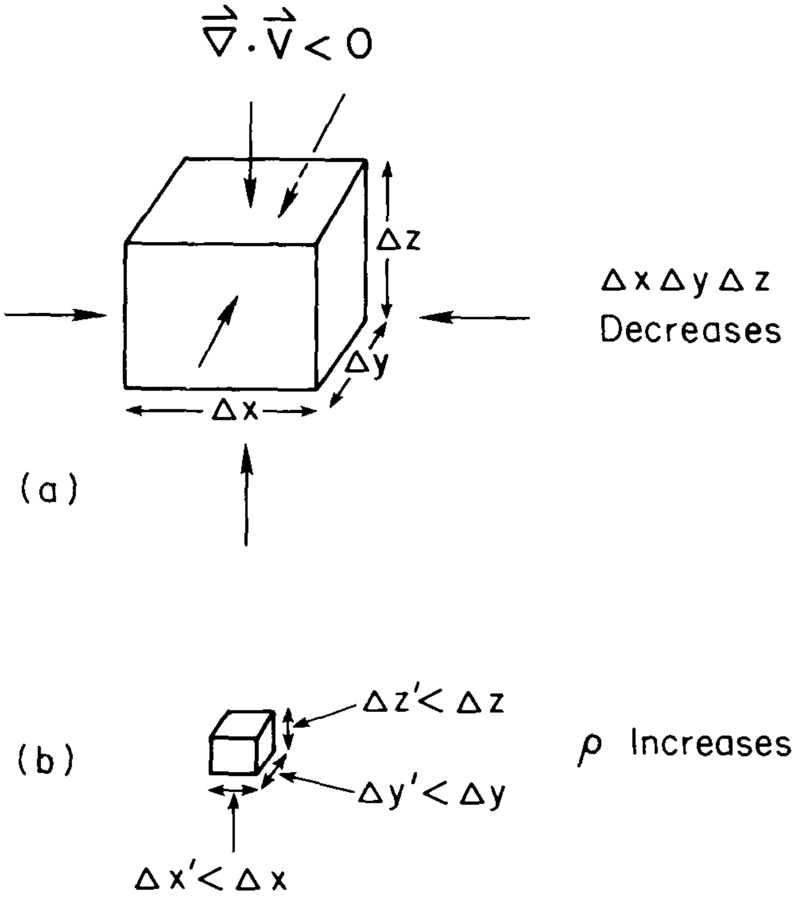
Balance Wind	Assumptions	Balance	
Geostrophic flow	Neglect friction and acceleration due to curvature	Pressure gradient force and Coriolis force	$\vec{V}_g = \vec{k} \times \frac{1}{\rho f} \nabla_z p$
Gradient wind	Neglect friction	Pressure gradient force, Coriolis force and centrifugal force	$\frac{-v_{gr}^2}{R_T} = -fv_g + fv_{gr}$
Cyclostrophic flow	Neglect friction and Coriolis force	Pressure gradient force and centrifugal force	$\frac{-v_d^2}{R_T} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$
Balance with Friction	All forces now included	Pressure gradient force, Coriolis force, centrifugal force and friction	$\frac{V_F^2}{R_T} + fV_F = fV_g + C_D V_F^2$

Continuity Equation

- The equation of continuity is given by:

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} = -\vec{\nabla} \cdot \vec{v}}$$

- Physical interpretation: if a volume having dimensions Δx, Δ y and Δ z experiences convergence, then the material volume decreases. However, since the amount of mass in a material volume remains constant, the density must increase



Source: Bluestein, 1992

- The continuity equation can also be expressed as:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \vec{v}$$

This is the flux form of the continuity equation – says that mass in a volume can change locally only through flux convergence or divergence

- If we assume that the atmosphere is incompressible then the density of the parcel does not change:

$$\frac{D\rho}{Dt} = 0$$

$$=> \quad \frac{D\rho}{Dt} = -\vec{\nabla} \cdot \vec{v} = 0$$

$$=> \quad \boxed{\vec{\nabla} \cdot \vec{v} = 0}$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

- This means that for an incompressible atmosphere the atmosphere is three-dimensionally nondivergent

- The incompressible assumption implies that convergence in one or two directions must be balanced by divergence in the other direction(s) and that mass is conserved.

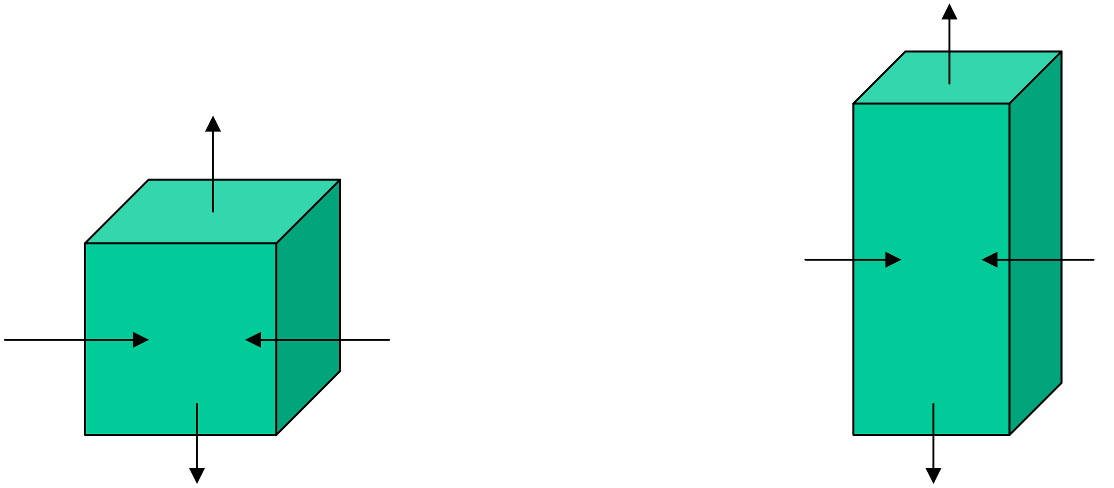
- The incompressible assumption is useful in helping to understand atmospheric systems that are not strongly dependant on compressibility. This approximation fails in strong thunderstorm updrafts, tornadoes etc

- For deep convection where the impacts of compressibility are important in the vertical, the following continuity equation is used:

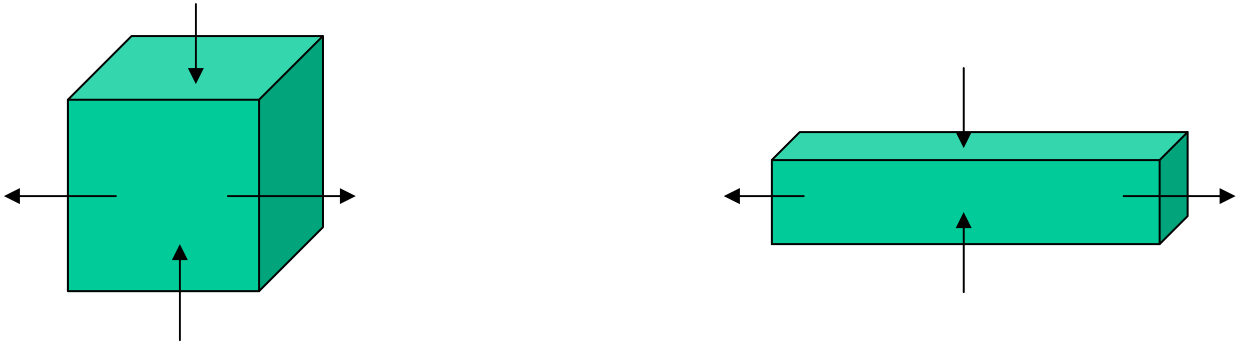
$$\vec{\nabla} \cdot [\rho(z)\vec{v}] = 0$$

where the base-state density $\rho(z)$ is a function of height only

- Assume that the atmosphere behaves as an incompressible fluid:



Horizontal convergence => Vertical stretching



Horizontal divergence => Vertical shrinking

We showed earlier that the continuity equation could be written in flux form as:

$$\frac{\partial\rho}{\partial t} = -\nabla \cdot \rho\vec{v}$$

$$\Rightarrow \frac{\partial\rho}{\partial t} = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)$$

If we assume that the atmosphere is incompressible:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

which can be written as:

$$\boxed{\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}_{\text{Horizontal Divergence}} = -\frac{\partial w}{\partial z}}$$

Level of Nondivergence

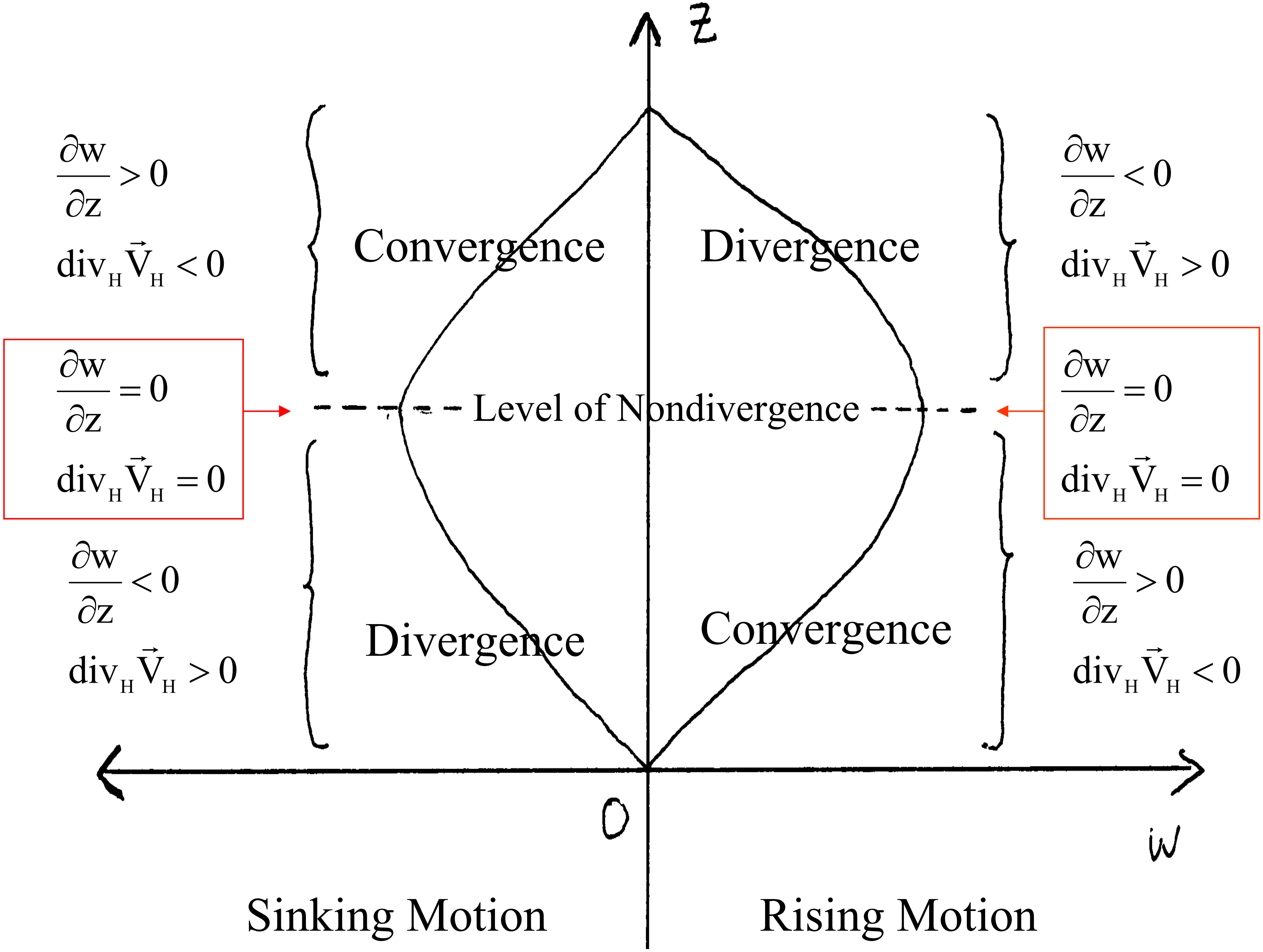
- Vertical velocity is constrained to be zero at the ground and at the tropopause
- If w is nonzero, its sign is often the same at all levels in a column of the troposphere => the sign of $\partial w/\partial z$ must reverse at some level. At this level:

  $$\frac{\partial w}{\partial z} = 0$$

  which from the continuity equation implies

  $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \mathrm{div}_H \vec{V}_H = 0$$

- This level is therefore called the level of nondivergence. It is typically found near 550-600mb.
- Rising motion above a level surface must be accompanied by convergence below and compensating divergence aloft.
- Similarly sinking motion must be accompanied by divergence below and convergence aloft



$$\frac{\partial w}{\partial z} > 0$$

$$\text{div}_H \vec{V}_H < 0$$

Convergence

Divergence

$$\frac{\partial w}{\partial z} < 0$$

$$\text{div}_H \vec{V}_H > 0$$

$$\frac{\partial w}{\partial z} = 0$$

$$\text{div}_H \vec{V}_H = 0$$

Level of Nondivergence

$$\frac{\partial w}{\partial z} = 0$$

$$\text{div}_H \vec{V}_H = 0$$

$$\frac{\partial w}{\partial z} < 0$$

$$\text{div}_H \vec{V}_H > 0$$

Divergence

Convergence

$$\frac{\partial w}{\partial z} > 0$$

$$\text{div}_H \vec{V}_H < 0$$

Sinking Motion

Rising Motion

Pressure Tendency Equation

The pressure at any height (z) is given by the weight of the air column above it :

$$-\int_{p}^{0} \mathrm{dp} = g\int_{z}^{\infty} \rho dz = \int_{0}^{p} dp = p$$

The pressure tendency at z is :

$$\frac{\partial \mathrm{p}}{\partial \mathrm{t}} = \frac{\partial}{\partial t}\left[g\int_{z}^{\infty} \rho dz \right] = g\int_{z}^{\infty} \frac{\partial \rho}{\partial t} dz$$

But we saw earlier that :

$$\frac{\partial \rho}{\partial \mathrm{t}} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z} = -div_H(\rho \vec{V}_H) - \frac{\partial(\rho w)}{\partial z}$$

where $\vec{\mathrm{V}}_{\mathrm{H}}$ is the horizontal wind vector and $\mathrm{div}_{\mathrm{H}}$ is the horizontal divergence

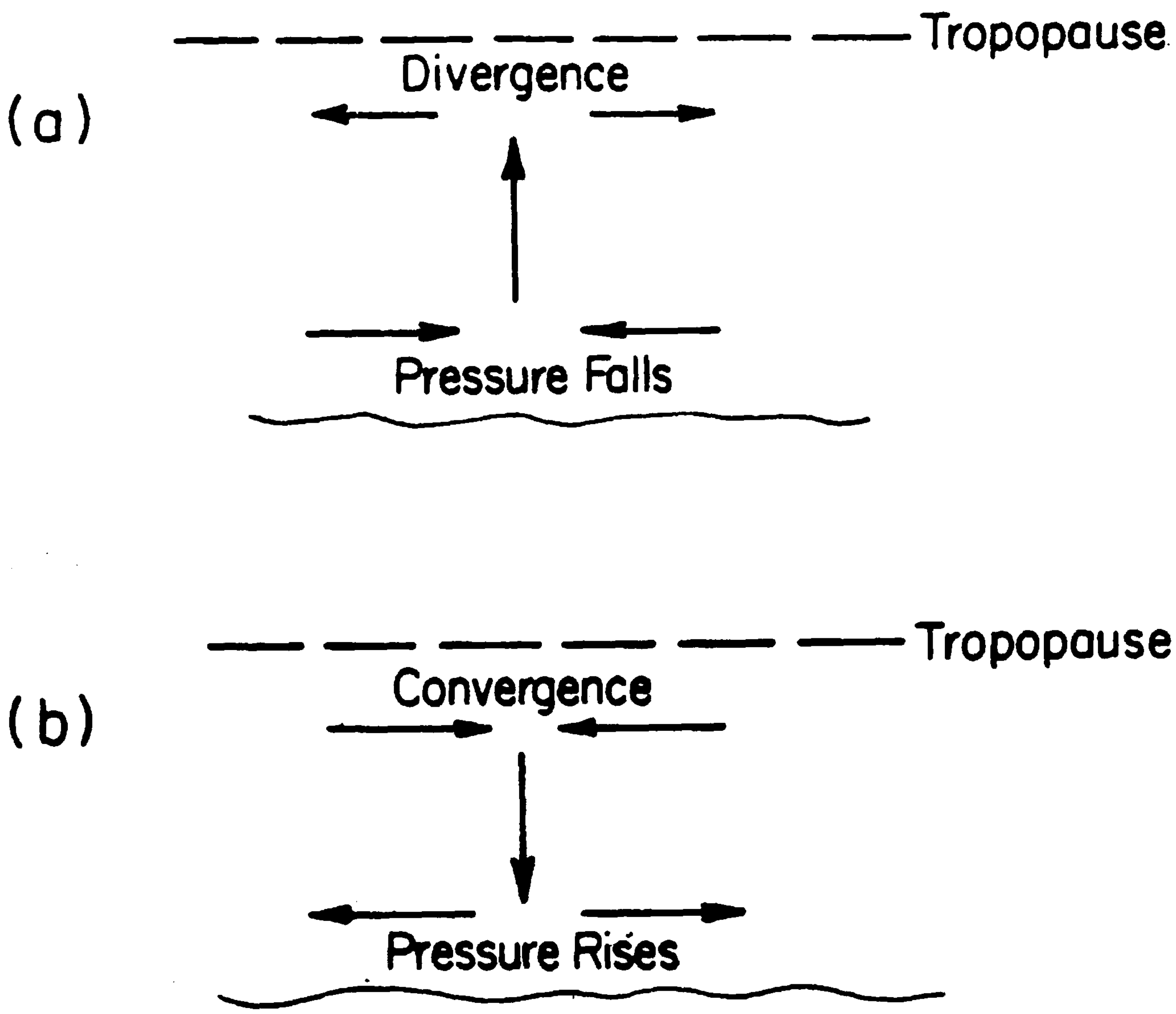
$$\left.\frac{\partial p}{\partial t}\right|_z = -g\int_z^\infty \mathrm{div}_H(\rho\vec{V}_H)dz - g\int_z^\infty \frac{\partial}{\partial z}(\rho w)dz$$

$$\text{For } g\int_z^\infty \frac{\partial}{\partial z}(\rho w)dz = \cancelto{0}{\rho w\big|_\infty} - \rho w\big|_z$$

$$\therefore \boxed{\left.\frac{\partial p}{\partial t}\right|_z = -g\int_z^\infty \mathrm{div}_H(\rho\vec{V}_H)dz + g\rho w\big|_z}$$

$$\text{At the surface: } (\rho w)_{z=0} = 0$$

$$=> \boxed{\left.\frac{\partial p}{\partial t}\right|_{z=0} = -g\int_{z=0}^\infty \mathrm{div}_H(\rho\vec{V}_H)dz}$$


Tropopause
Divergence
(a)
Pressure Falls
Tropopause
Convergence
(b)
Pressure Rises

Divergence and Vertical Motion

Using pressure coordinates and assuming an incompressible atmosphere the continuity equation can be written as :

$\frac{\partial\omega}{\partial p} = -\vec{\nabla}.\vec{V}$ where $\omega = \frac{Dp}{Dt}$ and the divergence refers to horizontal divergence

Integrating between pressure levels p_1 and p_2 :

$$\int_{p_1}^{p_2} \frac{\partial\omega}{\partial p} dp = -\int_{p_1}^{p_2} \vec{\nabla}.\vec{V} dp$$

A good approximation to the left hand side is :

$d\omega \approx (\partial\omega/\partial p)dp$ since the other partial derivatives are typically

much smaller than $\partial\omega/\partial p$

The integral on the right hand side integrates to :

$$-\overline{\vec{\nabla}.\vec{V}}(p_2 - p_1)$$

providing the correct vertical average divergence is selected. In the

case of constant $\vec{\nabla}.\vec{V}$ or linearly varying $\vec{\nabla}.\vec{V}$, the value of $\vec{\nabla}.\vec{V}$ in the

middle of the interval is equal to the average divergence $\overline{\vec{\nabla}.\vec{V}}$

So we now have:

$$\boxed{\omega_2 - \omega_1 = -\overline{\vec{\nabla}.\vec{V}}(p_2 - p_1)}$$ where ω_1 occurs at p_1 and ω_2 at p_2

Example: consider the layer between 1000 and 700mb then

$\omega_2 = \omega_{700}$ and $\mathrm{p}_2 = 700$ mb

$\omega_1 = \omega_{1000}$ and $\mathrm{p}_1 = 1000$ mb

For our purposes the divergence at 850 mb can be assumed equal to the average divergence in the layer between 1000 and 700 mb.

Another typical assumption is that the vertical motion at 1000 mb is much less than that at 700mb and is thus neglected. With these assumptions we get the following:

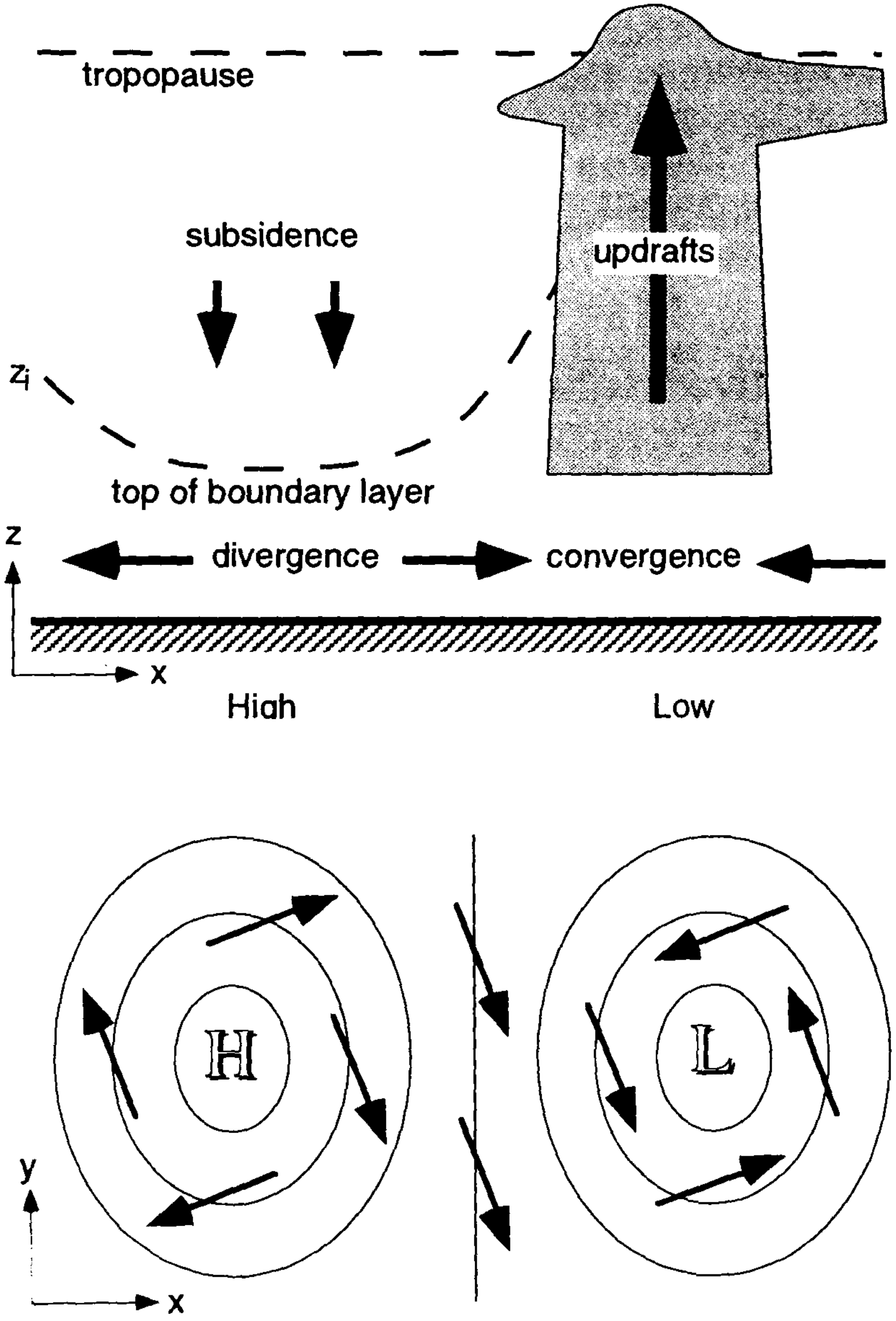
$$\omega_{700} = \vec{\nabla}.\vec{V}_{850}\Delta p \qquad \text{where } \Delta \mathrm{p} = \mathrm{p}_1 - p_2 = 300mb$$

Therefore convergence at 850mb ($\vec{\nabla}.\vec{V}_{850} < 0) \Rightarrow \omega_{700} < 0 \Rightarrow$ rising motion

divergence at 850mb ($\vec{\nabla}.\vec{V}_{850} > 0) \Rightarrow \omega_{700} > 0 \Rightarrow$ sinking motion

- Winds around high pressure systems diverge. Conservation of air mass requires subsidence over the high to replace the horizontally diverging air

- Similarly horizontally converging air around low pressure systems are associated with upward motion



Source: Stull, 2000

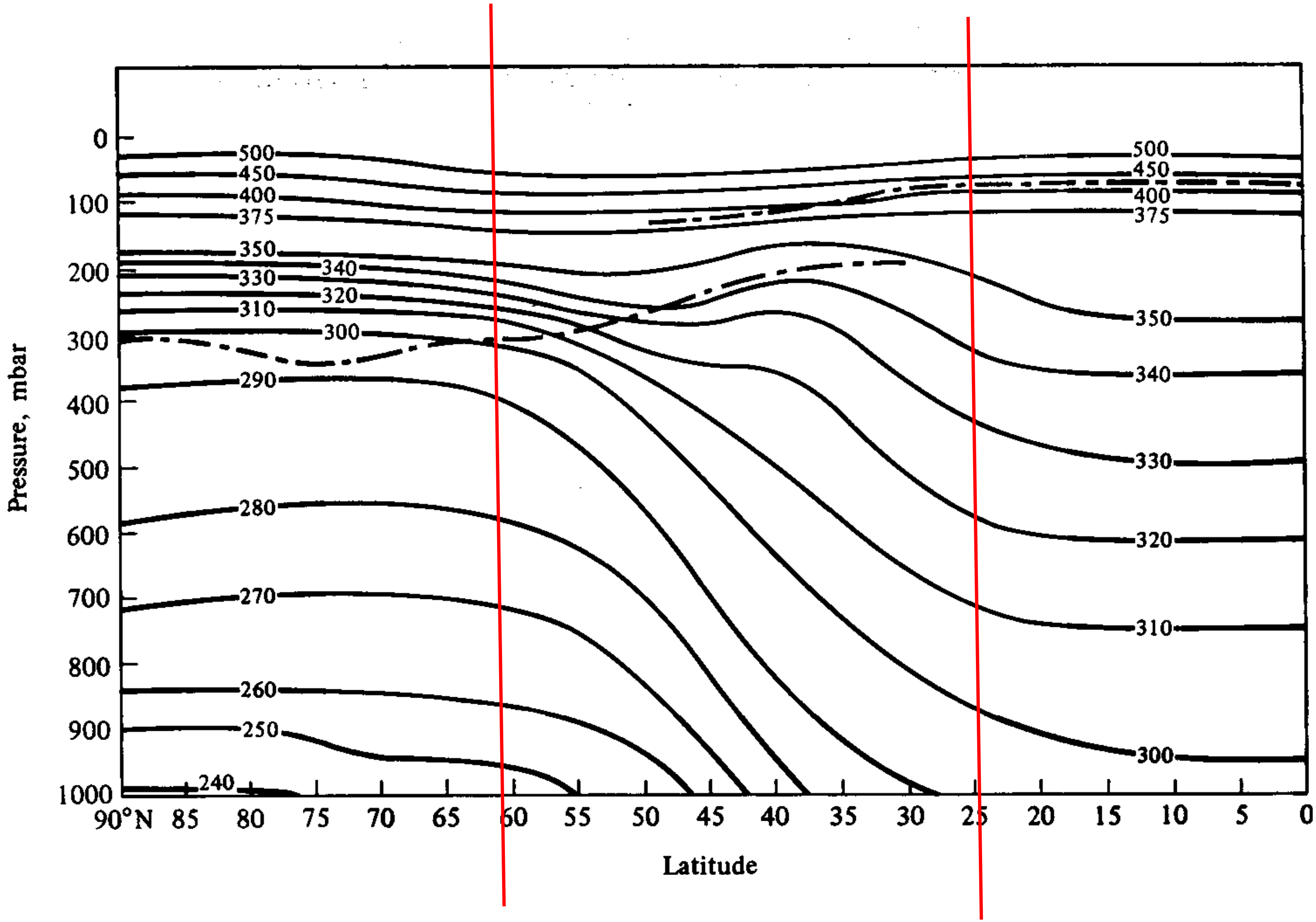
Baroclinity (Baroclinicity) and Barotropy

- Baroclinity (or Baroclinicity): The state of stratification in a fluid in which surfaces of constant pressure (isobaric) intersect surfaces of constant density (isoteric)
- Barotropy: The state of a fluid in which surfaces of constant density (or temperature) are coincident with surfaces of constant pressure; it is the state of zero baroclinity
- When there are temperature variations on an isobaric surface the atmosphere is said to be baroclinic. If there are no temperature variations the atmosphere is said to be barotropic

Barotropic Atmosphere	**Baroclinic Atmosphere**
ρ and p surfaces coincide	ρ and p surfaces intersect
p and T surfaces coincide	p and T surfaces intersect
p and θ surfaces coincide	p and θ surfaces intersect
No geostrophic wind shear	Geostrophic wind shear
No large-scale w	Large-scale w

Examples of baroclinic systems:

- Cold fronts
- Sea breezes
- Mountain slope flows



Barotropic Baroclinic Barotropic

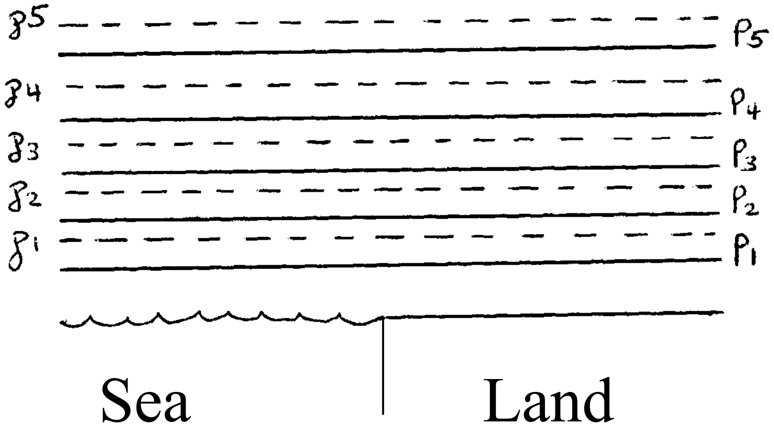
Barotropic Atmosphere

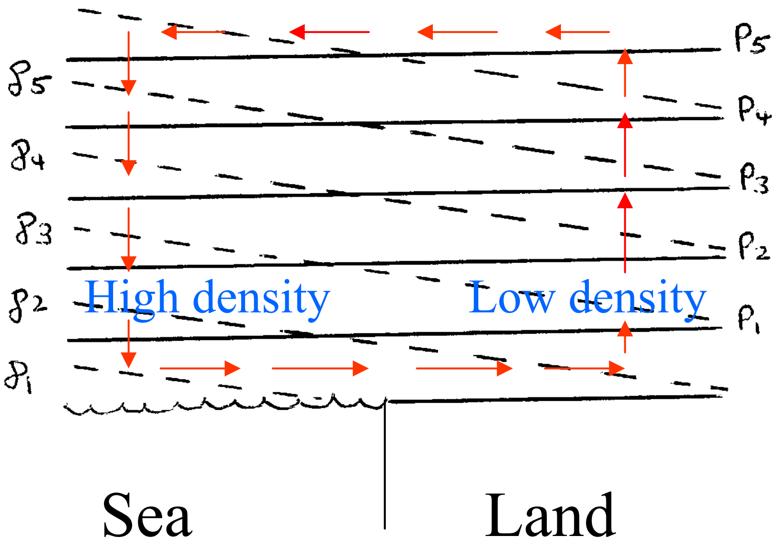
- eg: sea-breeze during early morning
- Pressure and density isolines are parallel / coincident
- No circulation

Baroclinic Atmosphere

- eg: sea-breeze in the afternoon
- Air over the land heats up more rapidly than that over water
- Pressure and density isolines intersect
- Circulation develops
- The lighter fluid over land “feels” the same pressure gradient force as that over the ocean – the lighter fluid will tend to rise more rapidly resulting in a net counterclockwise circulation

$P_1 > P_2 > \ldots > P_5$ and $\rho_1 > \rho_2 > \ldots . > \rho_5$





Vorticity

- Vorticity: A vector measure of the rotation in a fluid, and is defined mathematically as the curl of the velocity:

$$\boxed{\vec{\omega} = \nabla \times \vec{V}}$$

- The vorticity of a solid rotation is twice the angular velocity vector
- In meteorology, vorticity usually refers to the vertical component of the vorticity

Components of Vorticity

$$\vec{\omega} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k}$$

$$= \xi \, \vec{i} + \eta \, \vec{j} + \zeta \, \vec{k}$$

- The components of the vorticity ξ (xi), η (eta) and ζ (zeta) are measures of the spin about the x, y and z axes.

Relative and Absolute Vorticity

- Relative vorticity (or local vorticity) (ω):
  - the vorticity as measured in a system of coordinates fixed on the earth's surface
  - curl of the relative velocity

$$\boxed{\vec{\omega} = \vec{\nabla} \times \vec{V}}$$

- Absolute vorticity (ω_a)
  - the vorticity of a fluid particle determined with respect to an absolute coordinate system (takes into account the rotation of the earth)
  - curl of the absolute velocity

$$\boxed{\vec{\omega}_a = \vec{\nabla} \times \vec{V}_a}$$

- The vertical component of the absolute vorticity vector (as defined above) is given by the sum of the vertical component of the vorticity with respect to the earth (the relative vorticity) and the vorticity due to the rotation of the earth (equal to the Coriolis parameter) *f*:

$$\zeta_a = \zeta + f$$

- The difference between absolute and relative vorticity is therefore simply the planetary vorticity ($f=2\Omega \sin \varphi$):

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \zeta_a = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f$$

- Regions of large positive (negative) ζ tend to develop in association with cyclonic storms in the Northern (Southern) hemisphere – the distribution of relative vorticity is therefore an excellent diagnostic tool for weather analysis
- Absolute vorticity tends to be conserved following the motion at midtropospheric levels – forms the basis to simple dynamical forecast schemes

Vorticity Equation

We are now going to use the equations of motion to derive an equation for the time rate of change of the vertical component of vorticity:

The vertical component of vorticity is given by :

$$\zeta = \frac{\partial \mathrm{v}}{\partial \mathrm{x}} - \frac{\partial u}{\partial y}$$

The horizontal equations of motion are given by :

$$\frac{\mathrm{Du}}{\mathrm{Dt}} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + F_x \qquad (1)$$

$$\frac{\mathrm{Dv}}{\mathrm{Dt}} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + F_y \qquad (2)$$

To get the vertical component we subtract the partial derivative of (1) with respect to y from the partial derivative of (2) with respect to x:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}-fu+F_y\right)-$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+fv+F_x\right)$$

$$=>\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$$

$$+\frac{\partial u}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$+v\frac{\partial f}{\partial y}-\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)-\frac{\partial F_y}{\partial x}+\frac{\partial F_x}{\partial y}=0$$

$$=>\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+v\frac{\partial f}{\partial y}=\frac{D}{Dt}(\zeta+f)=$$

$$-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)+\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)$$

The vorticity equation is therefore given by:

$$\boxed{\frac{D}{Dt}(\zeta+f)=-(\zeta+f)\overbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}^{\text{Divergence}}-\overbrace{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}^{\text{Tilting/ Twisting}}+\overbrace{\frac{1}{\rho^{2}}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}^{\text{Solenoidal/ Baroclinic}}+\overbrace{\left(\frac{\partial F_{y}}{\partial x}-\frac{\partial F_{x}}{\partial y}\right)}^{\text{Friction}}}$$

where we used:

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

and the fact that the Coriolis parameter depends only on y so that:

$$\frac{Df}{Dt}=v\frac{\partial f}{\partial y}$$

The vorticity equation states that the rate of change of the absolute vorticity following the motion is given by the sum of the divergence term, the tilting or twisting term, the solenoidal term and the friction term

Divergence Term

- Ignoring the tilting, baroclinic and frictional terms we can write the vorticity equation as:

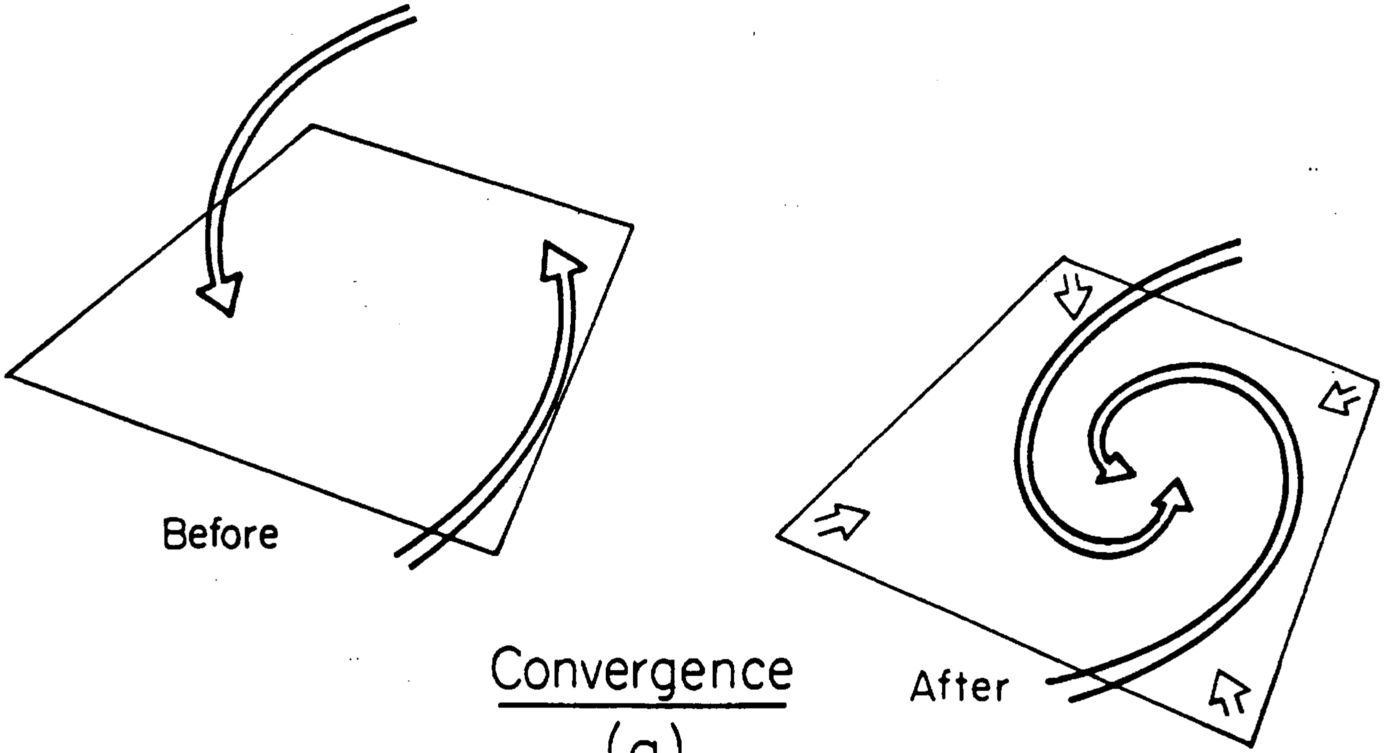
$$\frac{D}{Dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=-(\zeta+f)\mathrm{div}_H\vec{V}_H$$

where $\mathrm{div}_H\vec{V}_H$ is taken to mean $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$

- When $\mathrm{div}_H V_H < 0$ we have convergent flow => absolute vorticity increasing

- When $\mathrm{div}_H V_H > 0$ we have divergent flow => absolute vorticity decreasing

- Analogous to spinning ice skater pulling their arms in



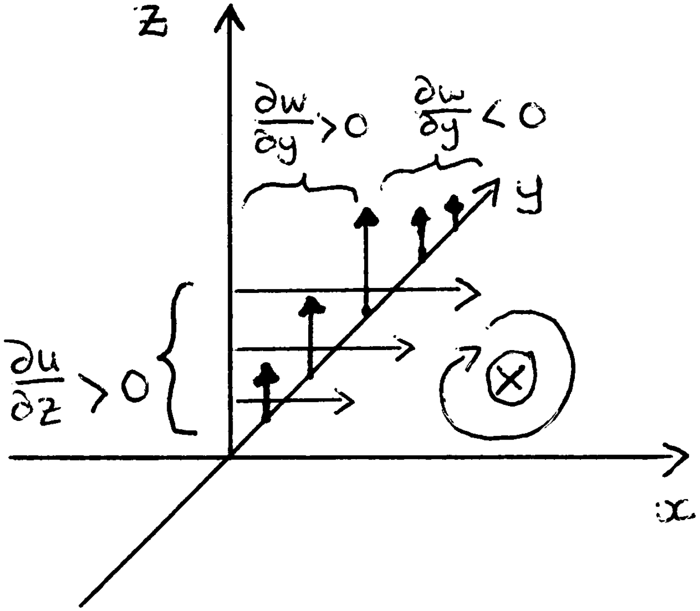
Convergence

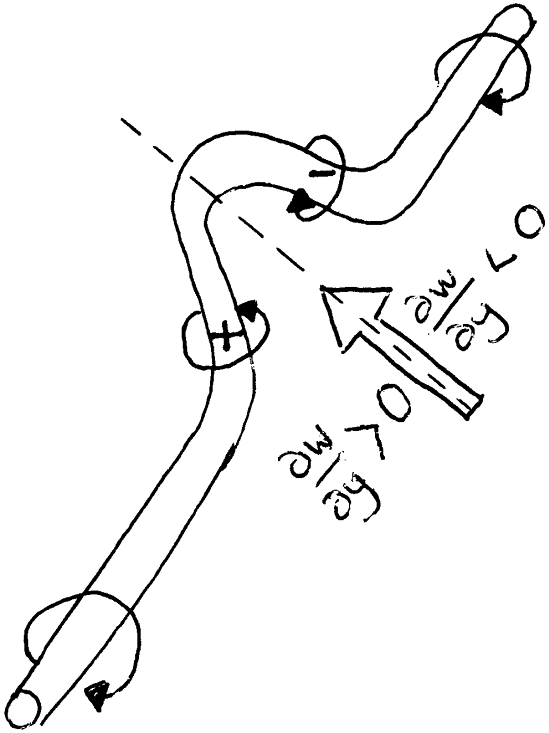
(a)

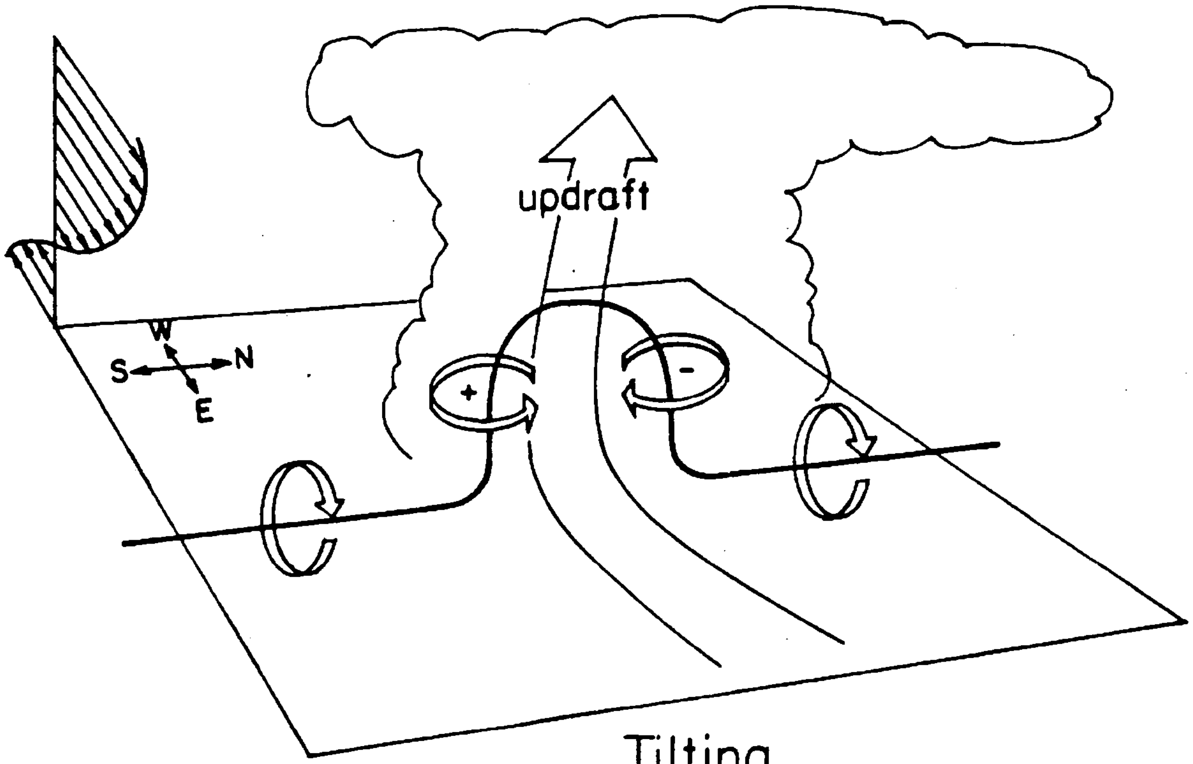
Tilting

$$\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}$$

Examining the first term:







Tilting
(b)

Baroclinic Term

The baroclinic term can be written in terms of temperature:

$$-\frac{1}{\rho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}-\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right)=-\frac{1}{\left(\frac{p}{Rt}\right)^2}\left(\frac{\partial p}{\partial x}\frac{\partial}{\partial y}\left(\frac{p}{RT}\right)-\frac{\partial p}{\partial y}\frac{\partial}{\partial x}\left(\frac{p}{RT}\right)\right)$$

$$=-\frac{RT^2}{p^2}\left(\frac{\partial p}{\partial x}\left[\frac{T\frac{\partial p}{\partial y}-p\frac{\partial T}{\partial y}}{T^2}\right]-\frac{\partial p}{\partial y}\left[\frac{T\frac{\partial p}{\partial x}-p\frac{\partial T}{\partial x}}{T^2}\right]\right)$$

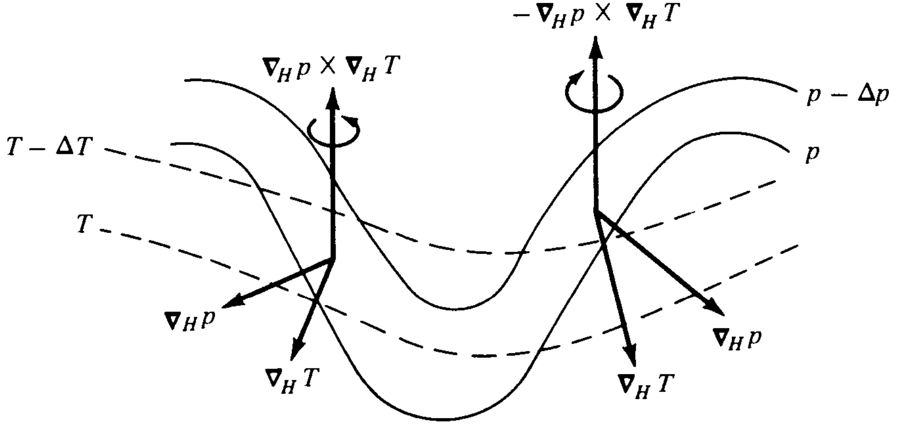
$$=-\frac{R}{p^2}\left(T\left(\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial p}{\partial x}\frac{\partial p}{\partial y}\right)+p\left(\frac{\partial p}{\partial y}\frac{\partial T}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial T}{\partial y}\right)\right)$$

$$=\frac{R}{p}\left(\frac{\partial p}{\partial x}\frac{\partial T}{\partial y}+\frac{\partial p}{\partial y}\frac{\partial T}{\partial x}\right)$$

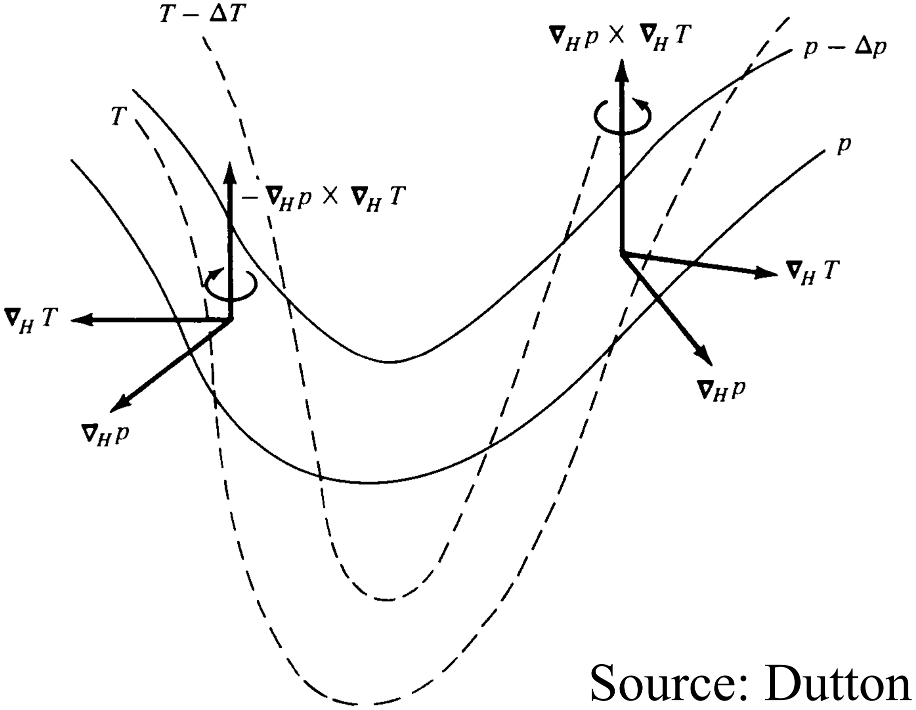
$$=\frac{R}{p}\vec{\nabla}_H p\times\vec{\nabla}_H T$$

Air moving through a wave pattern can acquire changing vorticity due to baroclinic stratification

Isotherms are of longer wavelength than the pressure wave – air moving through the wave will acquire cyclonic vorticity as it approaches the trough and intensification may be expected



Isothermal wave is of shorter wavelength and the air moving into the trough will be acquiring increasing anticyclonic vorticity so that the trough may be expected to become less pronounced



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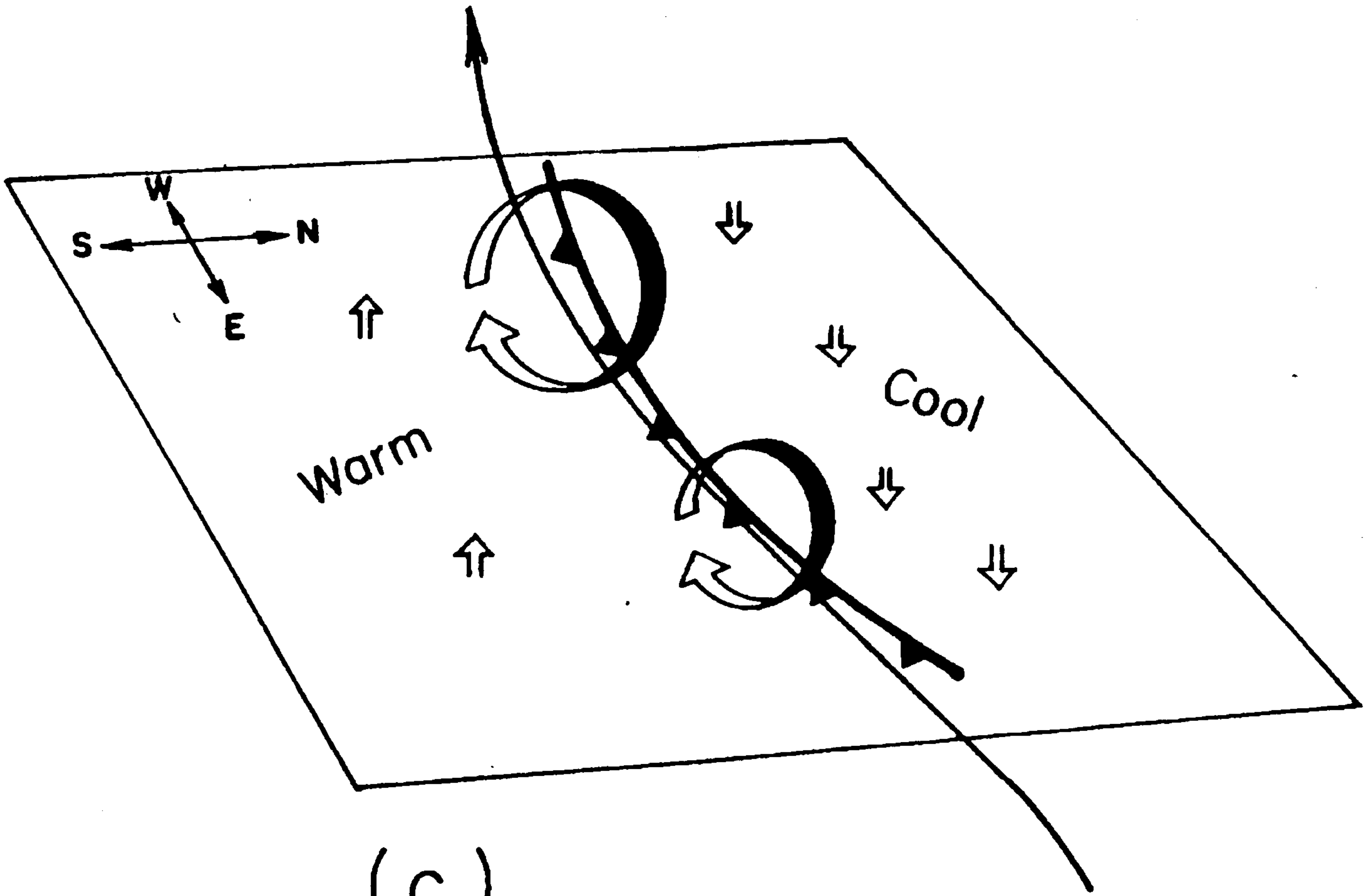


Baroclinicity

(c)

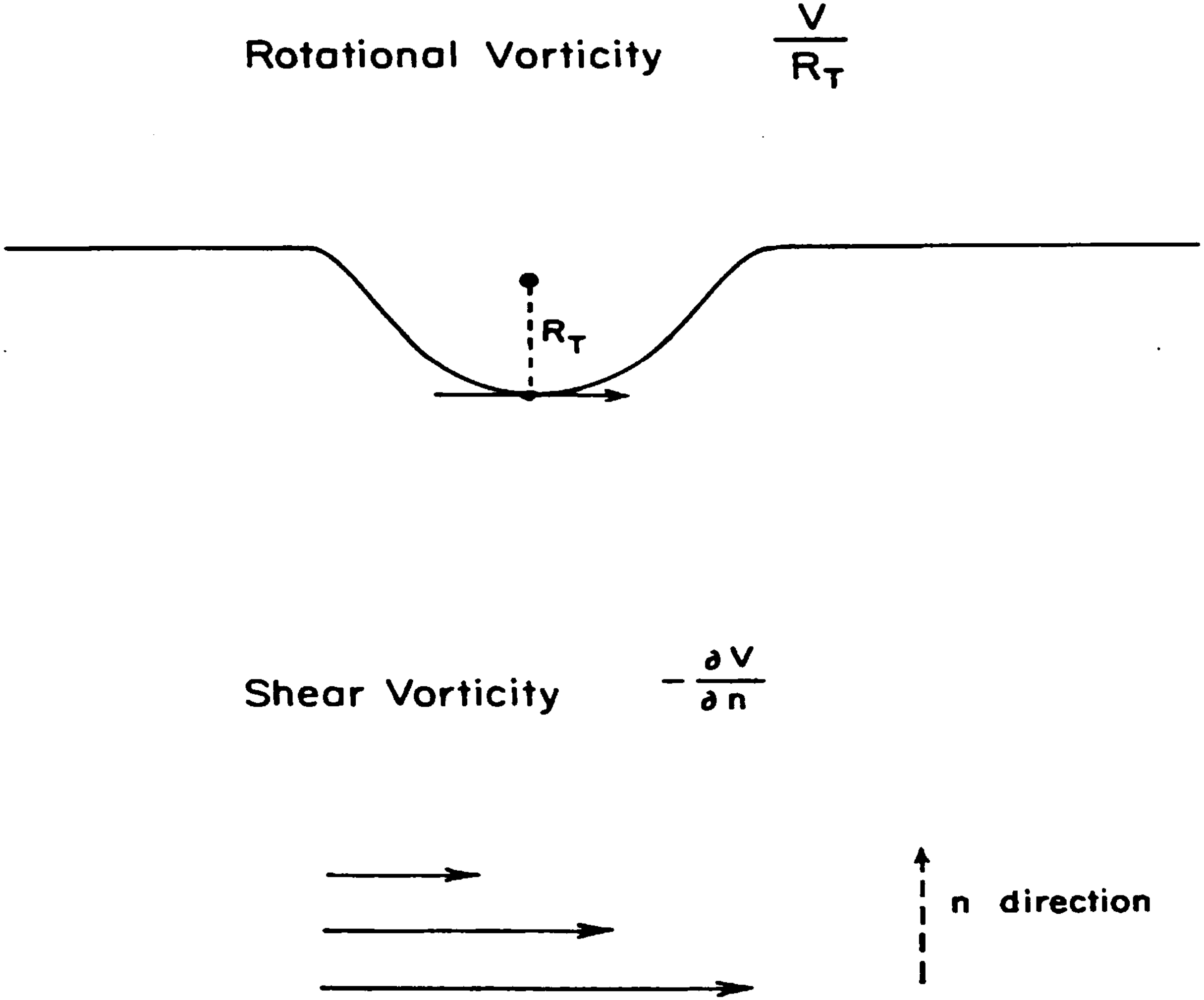
Relative Vorticity in Natural Coordinates

Relative vorticity can also be expressed in the so-called natural coordinates which are defined with respect to the parcel:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{V}{R_T} - \frac{\partial V}{\partial n}$$

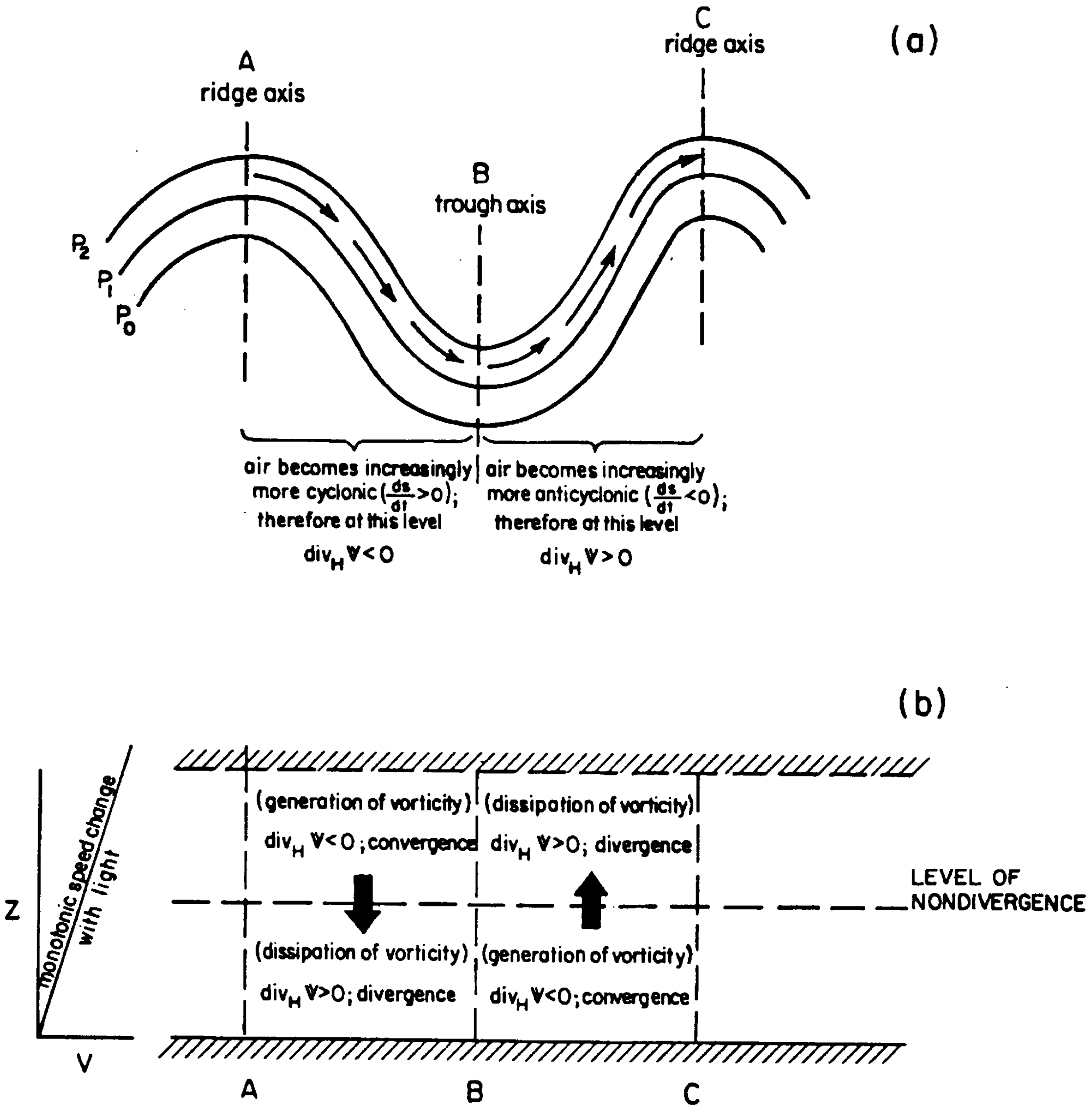
Rotational Vorticity: V/R_T represents the angular velocity of solid rotation of an air parcel about a vertical axis with radius of curvature R_T

Shear Vorticity: the lateral shear term, $-\partial V/\partial n$, represents the effective angular velocity of an air parcel produced by distortion due to horizontal velocity differences at its boundaries



Examples of rotational and shear cyclonic vorticity illustrated in natural coordinates.

- As wind speeds in the westerlies in the midlatitudes usually increase monotonically with height in the troposphere, the wind, and therefore the vorticity fields at the upper levels exert a major control on the synoptic vertical motion field as shown by $\mathrm{div}_H V \approx -\partial w/\partial z$
- We saw previously that to the extent that a parcel trajectory is in gradient wind balance the parcel will decelerate as it moves from a ridge crest into the trough, and accelerate as it moves from the trough to the ridge
- As $\mathrm{div}_H V \approx -\partial w/\partial z$ is generally a good approximation in the earth's troposphere, the vertical velocities seen in the next slide occur to conserve mass

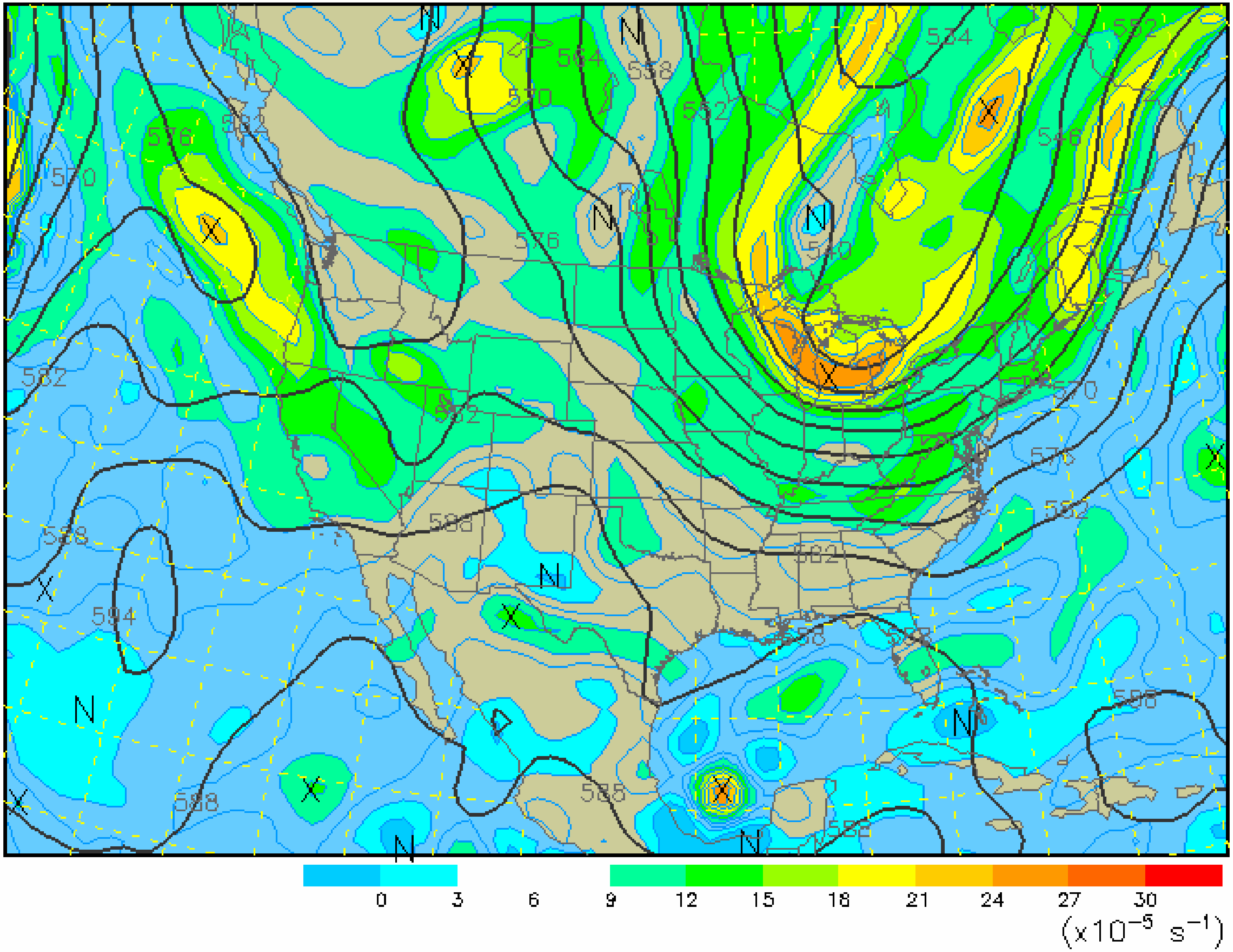


Schematic illustration of the inferred change of vorticity and resultant motion (b) as an air parcel in gradient wind balance moves through a constant pressure gradient wind field in the upper troposphere given in (a).

500 mb Heights (dm) / Abs. Vorticity $(x10^{-5}\ s^{-1})$

Analysis valid 0000 UTC Thu 02 Oct 2003

Eta (00z 02 Oct)



Some Issues to Look At

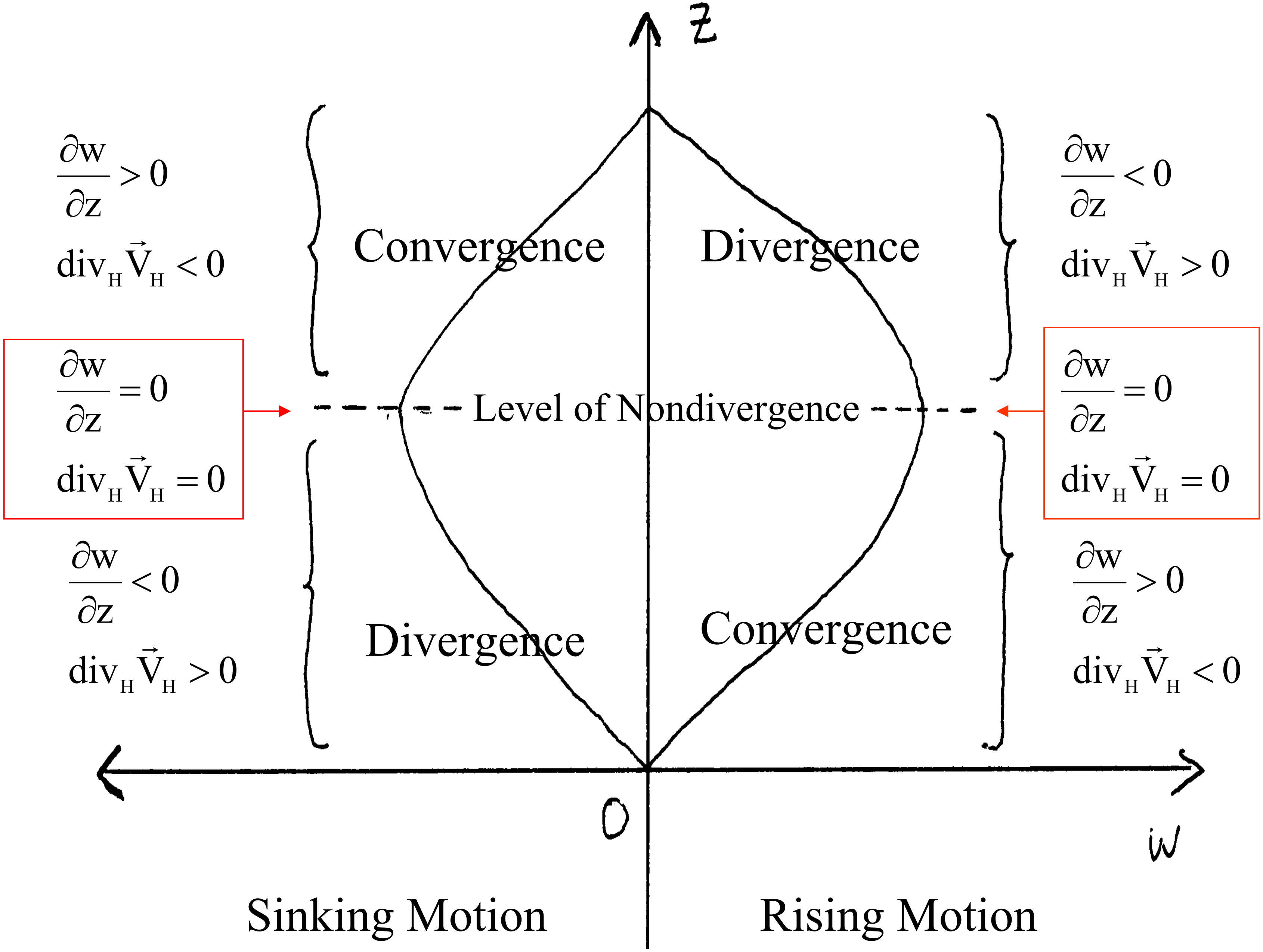
- Mid-Term Exam: Thursday October 16 12:10-2:10
- Greek letters names

  ξ (xi), η (eta) and ζ (zeta)
- Corrections made to zeta where need be
- Continuity equation interpretation
- Simple vorticity plots
- Pressure coordinates

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$$

- The continuity equation does NOT automatically imply that when the LHS > 0 (divergence) that this is associated with sinking motion (and visa versa for LHS < 0 and rising motion)
- To understand the relationship we need to know the profile of w – this includes the sign of w and whether w is increasing or decreasing with height
- Assumptions for determining relationship between conv/div and rising/sinking air:
  - w = 0 at the surface
  - w = 0 at the tropopause
  - w is the same sign in a column of air
    - Note: we could have w of different signs in the column – this would give us more than one level of nondivergence which is possible. However, on a synoptic scale, the assumption that w is of the same sign in a column is reasonable
- Knowledge of the w profile and its change with height then determines the level of nondivergence, and the relationship between w and divergence/convergence



$$\frac{\partial w}{\partial z} > 0$$

$$\mathrm{div}_H \vec{V}_H < 0$$

Convergence

Divergence

$$\frac{\partial w}{\partial z} < 0$$

$$\mathrm{div}_H \vec{V}_H > 0$$

$$\frac{\partial w}{\partial z} = 0$$

$$\mathrm{div}_H \vec{V}_H = 0$$

Level of Nondivergence

$$\frac{\partial w}{\partial z} = 0$$

$$\mathrm{div}_H \vec{V}_H = 0$$

$$\frac{\partial w}{\partial z} < 0$$

$$\mathrm{div}_H \vec{V}_H > 0$$

Divergence

Convergence

$$\frac{\partial w}{\partial z} > 0$$

$$\mathrm{div}_H \vec{V}_H < 0$$

Sinking Motion

Rising Motion

Vertical Vorticity Plots

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Options :

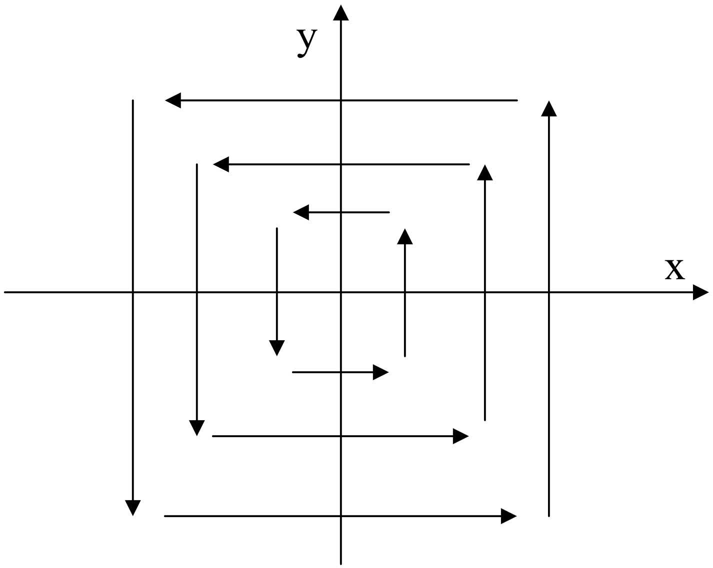
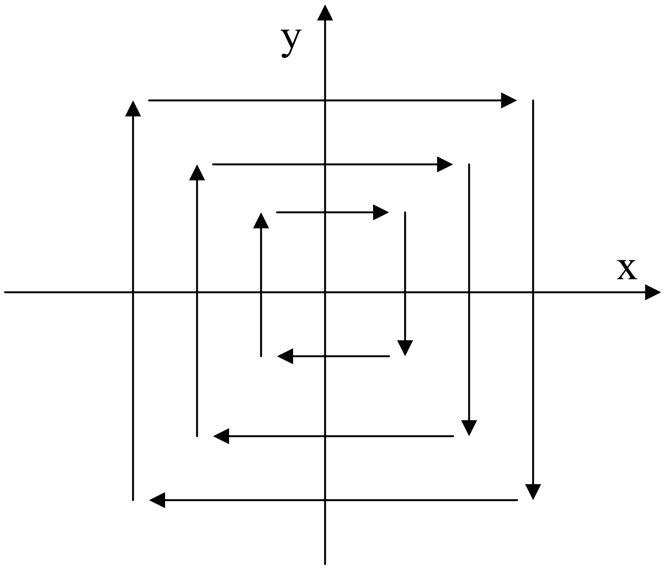
1) $\frac{\partial v}{\partial x} > 0$ and $\frac{\partial u}{\partial y} < 0$

2) $\frac{\partial v}{\partial x} < 0$ and $\frac{\partial u}{\partial y} > 0$

3) $\frac{\partial v}{\partial x} > 0$ and $\frac{\partial u}{\partial y} > 0$

4) $\frac{\partial v}{\partial x} < 0$ and $\frac{\partial u}{\partial y} < 0$

- Options 1 and 2 are ROTATIONAL motion
- Options 3 and 4 are called STRAINING motion

Positive Vorticity	Negative Vorticity
$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} > 0$	$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} < 0$
$\frac{\partial v}{\partial x} > 0, \quad \frac{\partial u}{\partial y} < 0$	$\frac{\partial v}{\partial x} < 0, \quad \frac{\partial u}{\partial y} > 0$
	
Cyclones (Northern Hemisphere) Anticyclones (Southern Hemisphere)	Anticyclones (Northern Hemisphere) Cyclones (Southern Hemisphere)

Primitive Equations in Isobaric Coordinates

1) Material Derivative

- Cartesian Coordinates: x, y, z, t

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\text{where } \; w = \frac{Dz}{Dt}$$

- Isobaric Coordinates: x, y, p, t

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}$$

$$\text{where } \omega = \frac{Dp}{Dt}$$

2) Pressure Gradient Term

- Any scalar φ can be represented in either coordinate system and the value of φ at a point (x,y,z) is the value of φ in the pressure coordinate system at the point (x,y,p) where z = z(x,y,p,t)

$$\phi(x,y,p,t) = \phi(x,y,z(x,y,p,t),t)$$

Using the chain rule:

$$\left.\frac{\partial\phi}{\partial x}\right|_{y,p,t} = \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial\phi}{\partial t}\frac{\partial t}{\partial x}$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{y,p,t} = \left.\frac{\partial\phi}{\partial x}\right|_{y,z,t} + \left.\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x}\right|_{y,p,t}$$

Similarly

$$\left.\frac{\partial\phi}{\partial y}\right|_{x,p,t} = \left.\frac{\partial\phi}{\partial y}\right|_{x,z,t} + \left.\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x}\right|_{x,p,t}$$

$$\left.\frac{\partial\phi}{\partial x}\right|_{y,p,t} = \left.\frac{\partial\phi}{\partial x}\right|_{y,z,t} + \left.\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x}\right|_{y,p,t}$$

$$\left.\frac{\partial\phi}{\partial y}\right|_{x,p,t} = \left.\frac{\partial\phi}{\partial y}\right|_{x,z,t} + \left.\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x}\right|_{x,p,t}$$

If we assume that ϕ is pressure:

$$\left.\frac{\partial p}{\partial x}\right|_{y,p,t} = \left.\frac{\partial p}{\partial x}\right|_{y,z,t} + \left.\frac{\partial p}{\partial z}\frac{\partial z}{\partial x}\right|_{y,p,t}$$

$$-\left.\frac{\partial p}{\partial x}\right|_{z} = \left.\frac{\partial p}{\partial z}\frac{\partial z}{\partial x}\right|_{p}$$

Assuming hydrostatic balance: $\frac{\partial p}{\partial z} = -\rho g$

$$-\left.\frac{\partial p}{\partial x}\right|_{z} = (-\rho g)\left.\frac{\partial z}{\partial x}\right|_{p}$$

Using $\Phi = gz$ where Φ is the geopotential

$$\boxed{-\frac{1}{\rho}\left.\frac{\partial p}{\partial x}\right|_{z} = -\left.\frac{\partial\Phi}{\partial x}\right|_{p}}$$

Similarly

$$\boxed{-\frac{1}{\rho}\left.\frac{\partial p}{\partial y}\right|_{z} = -\left.\frac{\partial\Phi}{\partial y}\right|_{p}}$$

Now we assumed hydrostatic balance:

$$\boxed{\frac{\partial z}{\partial p} = \frac{-1}{\rho g} \quad \text{or} \quad \frac{\partial\Phi}{\partial p} = -\alpha}$$

Therefore the frictionless equations of motion in isobaric coordinates can be written as:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} = -\frac{\partial\Phi}{\partial x} + fv$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} = -\frac{\partial\Phi}{\partial y} - fu$$

$$\frac{\partial\Phi}{\partial p} = -\alpha$$

3) Continuity Equation

It can be shown (see Holton pg 59-60) that the continuity equation in isobaric coordinates is given by:

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0}$$

It should be noted that the continuity equation does not have any reference to the density field, nor does it involve any time derivatives. This is one of the main advantages for using the isobaric coordinate system.

The frictionless primitive equations in isobaric coordinates are therefore given by :

$$\frac{\partial \vec{v}_H}{\partial t} + \vec{v}_H \cdot \nabla_p \vec{v}_H + \omega_p \frac{\partial \vec{v}_H}{\partial p} = -\nabla_p \Phi - f\vec{k} \times \vec{v}_H$$

$$\frac{\partial \Phi}{\partial p} = -\alpha$$

$$(\nabla \cdot \vec{v})_p = 0$$

where the subscript p refers to isobaric coordinates, $\vec{v}_H$ refers to the horizontal velocity vector, and $\nabla_p = \vec{i}\left(\frac{\partial}{\partial x}\right)_p + \vec{j}\left(\frac{\partial}{\partial y}\right)_p$ here

The Omega Equation

- It is useful to combine the vorticity equation and the first law of thermodynamics into a single equation that describes the vertical motion above the surface associated with extratropical cyclones and other types of synoptic weather features
- This equation, which we will now derive, is called the OMEGA equation

- Step 1: Derive vorticity equation on a constant pressure surface:
  - Proceed as we did before to form the vorticity equation by subtracting ∂/∂y of the u momentum equation in isobaric coordinates from ∂/∂x of the v momentum equation in isobaric coordinates
  - Neglecting the tilting and friction terms
  - We get:

$$\frac{\partial \zeta_p}{\partial t} + \vec{V} \cdot \nabla_p (\zeta_p + f) = (\zeta_p + f) \frac{\partial \omega}{\partial p} \qquad (1)$$

where ζ_p is the relative vorticity on a constant pressure surface,

$$\nabla_p = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} \text{ and } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \text{ has been replaced by } -\frac{\partial \omega}{\partial p} \qquad (2)$$

- Step 2: Form a geostrophic vorticity:
  - Using the geostrophic wind relation:

$$\vec{v}_g = \frac{g}{f}\vec{k} \times \nabla_p z \qquad (3)$$

we can obtain the geostrophic vorticity:

$$\vec{\nabla} \times \vec{v}_g = \frac{g}{f}\nabla_p^2 z = \zeta_g \qquad (4)$$

- From (4) the vorticity can be estimated from the curvature of the height contours on a constant pressure analysis
- Substituting (4) into (1), where ζ_p is set equal to ζ_g gives:

$$\frac{\partial}{\partial t}\frac{g}{f}\vec{\nabla}_p^2 z + \vec{v}\cdot\vec{\nabla}_p\left(\zeta_g + f\right) = \left(\zeta_g + f\right)\frac{\partial\omega}{\partial p} \qquad (5)$$

- Step 3: Include thermodynamics using the First Law of Thermodynamics

$$\frac{dq}{T} = c_p\frac{dT}{T} - \frac{\alpha}{T}dp \quad => \quad \frac{dq}{T} = c_p\frac{dT}{T} - R\frac{dp}{p}$$

$$\text{Now } \theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \quad => \quad \ln\theta = \ln T + \frac{R}{c_p}(\ln p_0 - \ln p)$$

$$=> \frac{c_p}{\theta}d\theta = \frac{c_p}{T}dT - \frac{R}{p}dp \quad => \quad \frac{dq}{T} = \frac{c_p}{\theta}d\theta$$

$$=> \frac{Q}{T} = c_p\frac{d\ln\theta}{dt} \qquad (6)$$

where Q represents changes in sensible heat of a parcel (diabatic effects). Q can include explicit synoptic-scale phase changes of water as represented by $-Ldw_s$, as well as radiative flux divergence and subsynoptic-scale phase changes of water due to cumulus clouds

- Equation (6) can be written as:

$$C_p\left[\frac{\partial\ln\theta}{\partial t}+u\frac{\partial\ln\theta}{\partial x}+v\frac{\partial\ln\theta}{\partial y}+\omega\frac{\partial\ln\theta}{\partial p}\right]=Q/T$$

Since using the gas law:

$$\theta=T\left[1000/p\right]^{R_d/C_p}=\frac{p\alpha}{R_d}\left(\frac{1000}{p}\right)^{R_d/C_p}$$

then

$$\frac{\partial\ln\theta}{\partial x}=\frac{\partial\ln\alpha}{\partial x};\frac{\partial\ln\theta}{\partial y}=\frac{\partial\ln\alpha}{\partial y};\frac{\partial\ln\theta}{\partial t}=\frac{\partial\ln\alpha}{\partial t}$$

on a constant pressure surface and

$$C_p\left[\frac{\partial\alpha}{\partial t}+u\frac{\partial\alpha}{\partial x}+v\frac{\partial\alpha}{\partial y}+\omega\alpha\frac{\partial\ln\theta}{\partial p}\right]=\frac{\alpha}{T}Q.$$

From the hydrostatic relation in a pressure coordinate framework (i.e., $\frac{\partial z}{\partial p} = -\alpha/g$):

$$\alpha = -g\frac{\partial z}{\partial p}$$

so that the above can also be written as:

$$C_p\left[-\frac{\partial}{\partial t}\left(g\frac{\partial z}{\partial p}\right) - u\frac{\partial}{\partial x}\left(g\frac{\partial z}{\partial p}\right) - v\frac{\partial}{\partial y}\left(g\frac{\partial z}{\partial p}\right) + \omega\alpha\frac{\partial \ln\theta}{\partial p}\right] = \frac{\alpha}{T}Q)$$

By convention:

$$\sigma = -\alpha\frac{\partial \ln\theta}{\partial p} = g\frac{\partial z}{\partial p}\frac{\partial \ln\theta}{\partial p}$$

is defined so that the above becomes, after rearranging:

$$\frac{\partial}{\partial t}\left(-g\frac{\partial z}{\partial p}\right) - \vec{V}\cdot\vec{\nabla}_p\left(g\frac{\partial z}{\partial p}\right) - \omega\sigma = \frac{\alpha}{C_pT}Q = \frac{R_d}{pC_p}Q.$$

Performing the operation $\partial/\partial p$: on (5)

$$g\nabla_p^2\frac{\partial}{\partial t}\frac{\partial z}{\partial p}+f\frac{\partial}{\partial p}\left[\vec{V}\cdot\nabla_p\left(\xi_g+f\right)\right]=f\left(f+\xi_g\right)\frac{\partial^2\omega}{\partial p^2};$$

performing the operation ∇_p^2 and assuming that σ is a function of pressure only yields:

$$-g\nabla_p^2\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial p}\right)-\nabla_p^2\left[\vec{V}\cdot\nabla_p\left(g\frac{\partial z}{\partial p}\right)\right]-\sigma\nabla_p^2\omega=\frac{R_d}{pC_p}\nabla_p^2Q.$$

Adding the last two equations produces:

$$f\frac{\partial}{\partial p}\left[\vec{V}\cdot\nabla_p\left(\xi+f\right)\right]-\nabla_p^2\left[\vec{V}\cdot\nabla_p\left(g\frac{\partial z}{\partial p}\right)\right]-\sigma\nabla_p^2\omega=\frac{R_d}{C_pp}\nabla_p^2Q+f\left(f+\xi_g\right)\frac{\partial^2\omega}{\partial p^2}.$$

Since $\partial z/\partial p = -\alpha/g = -RT/gp$, this relation can also be written as:

$$\boxed{\sigma\nabla_p^2\omega + f\left(f+\xi_g\right)\frac{\partial^2\omega}{\partial p^2} = f\frac{\partial}{\partial p}\left[\vec{V}\cdot\vec{\nabla}_p\left(\xi_g+f\right)\right] + \frac{R_d}{p}\nabla_p^2\left[\vec{V}\cdot\vec{\nabla}_p T\right] - \frac{R_d}{C_p p}\nabla_p^2 Q.} \tag{7}$$

This equation is called the *Omega equation* and represents a diagnostic second order differential equation for $\frac{dp}{dt}$.

The three terms on the right side represent the following:

$\frac{\partial}{\partial p}\left[\vec{V}\cdot\vec{\nabla}_p\left(\xi_g+f\right)\right] \longrightarrow$ vertical variation of the advection of absolute vorticity on a constant pressure surface.

$\nabla_p^2\left[\vec{V}\cdot\vec{\nabla}_p T\right] \longrightarrow$ the curvature of the advection of temperature on a constant pressure surface.

$\nabla_p^2 Q \longrightarrow$ the curvature of diabatic heating on a constant pressure surface.

Notes

- The Omega equation is a second order diagnostic (only spatial derivatives) equation in ω
- It does not require information on the vorticity tendency as with the vorticity equation
- It does not require information on the temperature tendency
- However, the terms on the RHS employ higher-order derivatives than are used in other methods of vertical velocity estimation

To show the importance of the different terms in (7), note that the lefthand side of (7) is of the form $\vec{\nabla}^2\omega$ (although quantative solution is not easily obtained to (7) as the coefficient of $\vec{\nabla}_p^2\omega$ is different from $\partial^2\omega/\partial^2 p$). If $\vec{\nabla}^2\omega$ has a wave form:

$$\vec{\nabla}^2\omega = -k^2 A \sin kx$$

where A is a constant, and $k = 2\pi/L$ where L is the wavelength, then:

$$\omega \sim A \sin kx$$

therefore

$$\boxed{\vec{\nabla}^2\omega \sim -\omega}$$

Since

$$\omega = \frac{Dp}{Dt} = \frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} \cong w\frac{\partial p}{\partial z} = -w\rho g$$

for typical synoptic values of pressure tendency and pressure gradient, then:

$$\boxed{\omega \sim \text{-}w}$$

and therefore

$$\boxed{w \sim \vec{\nabla}^2\omega}$$

We now use these findings to interpret the three terms of the Omega equation more easily

First Term:Vorticity Advection

Using the relation between $\partial/\partial p$ and $\partial/\partial z$, and our observation that $\nabla^2\omega \sim w$,

$$w \sim \frac{\partial}{\partial p}\left[\vec{V}\cdot\vec{\nabla}_p\left(\xi_g+f\right)\right] \sim -\frac{\partial}{\partial z}\left[\vec{V}\cdot\vec{\nabla}_p\left(\xi_g+f\right)\right]$$

In most situations in the atmosphere, the vorticity advection is much smaller in the lower troposphere than in the middle and upper troposphere since $\vec{V}$ and ξ_g are usually smaller near the surface. We have shown that on the synoptic scale, cold air towards the poles requires that $\vec{V}$ becomes more positive with height.

Using this observation of the behavior of $\vec{V}$ and ξ_g with height:

$$w \sim -\vec{V}\cdot\vec{\nabla}_p\left(\xi_g+f\right)$$

In other words, vertical velocity is proportional to vorticity advection. Since upper-level vorticity patterns are usually geographically the same as at midtropospheric levels (since troughs and ridges are nearly vertical in the upper troposphere, the 500 mb level is generally chosen to estimate vorticity advection. This level is also close to the level of nondivergence in which creation or dissipation of relative vorticity is small, so that the conservation of absolute vorticity is a good approximation.

Thus for the Northern Hemisphere where $\xi_g > 0$ for cyclonic vorticity,

$w > 0$ if $-\vec{V} \cdot \vec{\nabla}_p (\xi_g + f) > 0$	positive vorticity advection (PVA)
$w < 0$ if $-\vec{V} \cdot \vec{\nabla}_p (\xi_g + f) < 0$	negative vorticity advection (NVA)

To generalize this concept to the southern hemisphere, PVA should be called cyclonic vorticity advection; NVA should be referred to as anticyclonic vorticity.

Second Term: Temperature Advection

The curvature of the advection of temperature on a constant pressure surface can be represented by :

$$\vec{\nabla}_{\mathrm{p}}^{2}\left[\vec{V}\cdot\vec{\nabla}_{p}T\right]\sim -k^{2}B\sin kx$$

where B is a constant. Therefore,

$$\vec{V}\cdot\vec{\nabla}_{p}T\sim B\sin kx$$

Since:

$$w\sim\nabla_{p}^{2}\left[\vec{V}\cdot\vec{\nabla}_{p}T\right]$$

then

$$w\sim -\vec{V}\cdot\vec{\nabla}_{p}T.$$

$$w \sim -\vec{V} \cdot \vec{\nabla}_p T.$$

Thus,

$w > 0$ if $-\vec{V} \cdot \vec{\nabla}_p T > 0$	warm advection
$w < 0$ if $-\vec{V} \cdot \vec{\nabla}_p T < 0$	cold advection

The 700 mb surface is often used to evaluate the temperature advection patterns since the gradients of temperature are often larger at this height than higher up and the winds are significant in speed. The 850 mb height can be used (when the terrain is low enough) although the values of $\vec{V}$ are often substantially smaller.

Third Term: Diabatic Heating

Finally, since $\nabla_p^2 Q \sim -k^2 \; C \sin kx$ can be assumed in this form, $w \sim -\nabla_p^2 Q$,and $Q \sim w$ results.

Therefore,

$w > 0$	diabatic heating
$w < 0$	diabatic cooling

An example of diabatic heating on the synoptic scale is deep cumulonimbus activity. An example of diabatic cooling is longwave radiative flux divergence

Summary

- The preceding analysis suggests the following relation between vertical motion, vorticity, temperature advection and diabatic heating:

w > 0	**w < 0**
Positive Vorticity Advection (PVA)	Negative Vorticity Advection (NVA)
Warm Advection	Cold Advection
Diabatic Heating	Diabatic Cooling

- When a combination of terms exist that separately would result in different signs of vertical motion (eg PVA with cold advection), the resultant vertical motion will depend on the relative magnitudes of the individual contributions
- Remember: this relation for vertical motion is only accurate as long as the assumptions used to derive the Omega equation are valid

Rules of Thumb for Synoptic Analyses

Vorticity Advection	• Evaluate at 500 mb
Temperature Advection	• Evaluate at 700 mb • At elevations near sea level, also evaluate at 850 mb
Diabatic Heating	• Contribution of major importance in synoptic weather patterns (especially cyclogenesis) are areas of deep cumulonimbus • Refer to geostationary satellite imagery for determination of locations of deep convection

General Notes

- Mid-term exam: Thursday October 16, 12:10 – 2:10
- Includes theory and lab applications, weighted more heavily toward theory
- Derivations are fair game although the following derivations will NOT be included: virtual temperature, enthalpy, vorticity equation, Omega equation, Q vector form of the Omega equation, and Petterssen’s equation. You must however understand the final form of the various equations, be able to interpret their terms and use these equations in explanations of various weather phenomena
- Bring questions to class next Tuesday

Omega Equation

$$\boxed{\sigma\nabla_p^2\omega+f\left(f+\xi_g\right)\frac{\partial^2\omega}{\partial p^2}=f\frac{\partial}{\partial p}\left[\vec{V}\cdot\vec{\nabla}_p\left(\xi_g+f\right)\right]+\frac{R_d}{p}\nabla_p^2\left[\vec{V}\cdot\vec{\nabla}_pT\right]-\frac{R_d}{C_pp}\nabla_p^2Q.}$$

The Q Vector

- Although the Omega Equation has 3 terms that are clearly interpreted as 3 separate physical processes, in practice there is often a significant amount of cancellation between the terms. Also they are not invariant under a Galilean transformation of the zonal coordinate (adding a constant mean zonal velocity will change the magnitude of each of the terms without changing the net forcing of vertical motion).
- As a result, an alternative form of the Omega equation, the Q-vector form, has been developed in which the forcing of the vertical motion is expressed in terms of the divergence of the horizontal vector forcing field
- The derivation of the Q-vector Omega equation from Cotton's notes follows. It is included for completeness and will become more meaningful once the associated approximations and assumptions have been covered in dynamics.

On the f plane the quasi-geostrophic prediction equations may be expressed simply as follow:

$$\frac{D_g u_g}{Dt} - f_0 v_a = 0 \tag{Q1}$$

$$\frac{D_g v_g}{Dt} + f_0 u_a = 0 \tag{Q2}$$

$$\frac{D_g T}{Dt} - S_\rho \omega = 0 \tag{Q3}$$

These are coupled by the thermal wind relationship

$$p\frac{\partial u_g}{\partial p} = \frac{R}{f_0}\frac{\partial T}{\partial y}, \quad p\frac{\partial v_g}{\partial p} = -\frac{R}{f_0}\frac{\partial T}{\partial x} \tag{Q4}$$

We now eliminate the time derivatives by first taking

$$p\frac{\partial}{\partial p}(Q1) - \frac{R}{f_0}\frac{\partial}{\partial y}(Q3)$$

to obtain

$$p\frac{\partial}{\partial p}\left[\frac{\partial u_g}{\partial t}+u_g\frac{\partial u_g}{\partial x}+v_g\frac{\partial u_g}{\partial y}-f_0v_a\right]-\frac{R}{f_0}\frac{\partial}{\partial y}\left[\frac{\partial T}{\partial t}+u_g\frac{\partial T}{\partial x}+v_g\frac{\partial T}{\partial y}-S_p\omega\right]=0$$

Using the chain rule of differential equations, this may be rewritten as

$$\frac{RS_p}{f_0}\frac{\partial\omega}{\partial y}-f_0p\frac{\partial v_a}{\partial p}=-\left(\frac{\partial}{\partial t}+u_g\frac{\partial}{\partial x}+v_g\frac{\partial}{\partial y}\right)\left(p\frac{\partial u_g}{\partial p}-\frac{R}{f_0}\frac{\partial T}{\partial y}\right)$$
$$-p\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right]+\frac{R}{f_0}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right]$$

But, by the thermal wind relation (Q4) the term in parenthesis on the right-hand side vanishes and

$$-p\left[\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x}+\frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right]=-\frac{R}{f_0}\left[\frac{\partial T}{\partial y}\frac{\partial u_g}{\partial x}-\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right]$$

Using these facts, plus the fact that

$$\partial u_g / \partial x + \partial v_g / \partial y = 0$$

we finally obtain the simplified form

$$\sigma \frac{\partial \omega}{\partial y} - f_0^2 \frac{\partial v_a}{\partial p} = -2Q_2 \qquad \text{(Q5)}$$

where

$$Q_2 \equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial y} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right] = -\frac{R}{p}\frac{\partial V_g}{\partial y} \cdot \nabla T$$

Similarly, if we take

$$p\frac{\partial}{\partial p}(Q2) + \frac{R}{f_0}\frac{\partial}{\partial x}(Q3)$$

followed by application of (Q4) we obtain

$$\sigma \frac{\partial \omega}{\partial x} - f_0^2 \frac{\partial u_a}{\partial p} = -2Q_1 \tag{Q6}$$

where

$$Q_1 \equiv -\frac{R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right] = -\frac{R}{p}\frac{\partial \mathbf{V}_g}{\partial x} \cdot \nabla T$$

If we now take $\partial\,(Q6)\,/\partial x + \partial\,(Q5)/\partial y$ and use the continuity equation

to eliminate the ageostrophic wind, we obtain the Q-vector form of the

omega equation:

$$\sigma \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -2\nabla \cdot Q$$

where

$$Q \equiv (Q_1, Q_2) = \left(-\frac{R}{p}\frac{\partial V_g}{\partial x} \cdot \nabla T, -\frac{R}{p}\frac{\partial V_g}{\partial y} \cdot \nabla T \right)$$

- The Q-vector form of the Omega equation is given by:

$$\sigma\nabla^2\omega + f_0{}^2\frac{\partial^2\omega}{\partial p^2} = -2\vec{\nabla}\cdot\vec{Q}$$

where

$$\vec{\mathrm{Q}} = \left(\mathrm{Q}_1, \mathrm{Q}_2\right) = \left(-\frac{R}{p}\frac{\partial\vec{V}_g}{\partial x}\cdot\nabla T, -\frac{R}{p}\frac{\partial\vec{V}_g}{\partial y}\cdot\nabla T\right)$$

and f_0 is a constant (f plane approximation)

- This equation shows that on the f plane vertical motion is forced only by the divergence of Q
- Unlike the traditional form of the Omega equation, the Q-vector form does not have forcing terms that partly cancel.
- The forcing of ω can be represented simply by the pattern of the Q vector
- It is evident from this form of the Omega equation that regions where Q is convergent (divergent) correspond to upward (downward) motion

Now we just saw that

$$\vec{Q}=\left(Q_1,Q_2\right)=\left(-\frac{R}{p}\frac{\partial\vec{V}_g}{\partial x}\cdot\nabla T,-\frac{R}{p}\frac{\partial\vec{V}_g}{\partial y}\cdot\nabla T\right)$$

$$=\left(\frac{-R}{p}\left[\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right],\frac{-R}{p}\left[\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x}+\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right]\right)$$

To understand what this means physically we place our x - axis parallel to the local isotherm with cold air on the left, then above expression can be simplified to :

$$\vec{Q}=\left(Q_1,Q_2\right)=\left(-\frac{R}{p}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y},-\frac{R}{p}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)=-\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x},\frac{\partial v_g}{\partial y}\right)$$

But $\frac{\partial u_g}{\partial x}=\frac{-\partial v_g}{\partial y}$ therefore

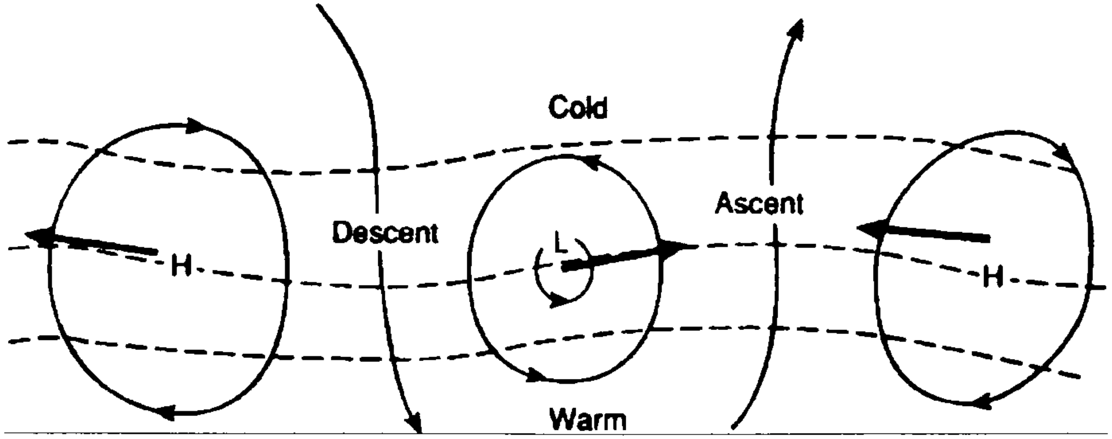
$$\vec{Q}=\left(Q_1,Q_2\right)=-\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x},\frac{-\partial u_g}{\partial x}\right)=-\frac{R}{p}\frac{\partial T}{\partial y}\left(\frac{\partial v_g}{\partial x}\vec{i}-\frac{\partial u_g}{\partial x}\vec{j}\right)$$

=> $$\vec{Q}=-\frac{R}{p}\left|\frac{\partial T}{\partial y}\right|\left(k\times\frac{\partial\vec{V}_g}{\partial x}\right)$$

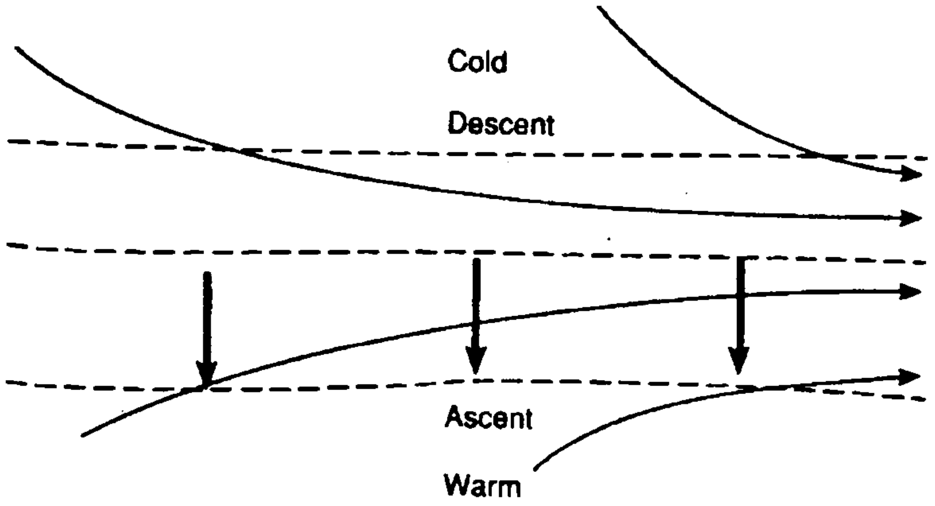
$$\vec{Q} = -\frac{R}{p}\left|\frac{\partial T}{\partial y}\right|\left(k \times \frac{\partial \vec{V}_g}{\partial x}\right)$$

- We can then get the Q-vector by determining the vectorial change of $\mathbf{V_g}$ along the isotherm (with cold air on the left), then rotating the resulting change vector by 90° clockwise, and then multiplying the resulting vector by $|\partial T/\partial y|$

- In the regions on the map where Q-vectors converge there is rising motion, and in the regions where the Q-vectors diverge there is sinking motion



Q vectors (bold arrow) for idealized pattern of isobars (solid) and isotherms (dashed) for a family of cyclones and anticyclones. (After Sanders and Hoskins, 1990).



Orientation of Q vectors (bold arrows) for confluent (jet entrance flow. Dashed lines are isotherms. (After Sanders and Hoskins, 1990).

Petterssen's Development Equation

- We now derive an equation that gives us information about the change of surface absolute vorticity
- If the vertical advection of absolute vorticity, the tilting term and the solenoidal term are ignored, then the vorticity equation can be written as:

$$\frac{\partial(\zeta_z + f)}{\partial t} + \vec{V}_H \cdot \vec{\nabla}_p (\zeta_z + f) = 0$$

where we assumed that the above equation is valid at the level of nondivergence (~ 500 mb)

$\vec{V}_H$ is the wind on the pressure surface

- Since, if the wind is in geostrophic balance:

$$\vec{V}_{H_{500}} = \vec{V}_{H_{SFC}} + \Delta\vec{V}_g$$

where $\Delta\vec{V}_g$ is the geostrophic wind shear. Thus,

$$(\xi_z + f)_{500} = (\xi_z + f)_{SFC} + (\xi_z + f)_T$$

since $\nabla \times \vec{V}_{H_{500}} = \left(\nabla \times \vec{V}_{H_{SFC}}\right) + \left(\nabla \times \Delta\vec{V}\right)$. We can write the vorticity equation as:

$$\frac{\partial(\xi_z + f)_{SFC}}{\partial t} = -\vec{V}_{H_{500}} \cdot \nabla_p(\xi_z + f)_{500} - \frac{\partial(\xi_z + f)_T}{\partial t}$$

From the thermal wind equation,

$$\nabla \times \Delta\vec{V}_g = \frac{g}{f}\nabla_p^2\left(\Delta z\right)$$

where $\Delta z = z_{500} - z_G$ with z_{500} the 500 mb height and z_G the surface elevation so that,

$$\frac{\partial(\xi_z + f)_T}{\partial t} = \frac{g}{f}\nabla_p^2\frac{\partial\left(\Delta z\right)}{\partial t}$$

We need an expression for $\frac{\partial(\Delta z)}{\partial t}$

We saw the following equation in our derivation of the Omega equation :

$$\frac{\partial}{\partial t}\left(-g\frac{\partial z}{\partial p}\right)-\vec{V}\cdot\vec{\nabla}_p\left(g\frac{\partial z}{\partial p}\right)-\omega\sigma=\frac{R_d}{pc_p}Q$$

Integrating between the surface pressure, p_{SFC}, and 500 mb yields, after rearranging:

$$-g\int_{p_{SFC}}^{500}\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial p}\right)dp=-g\frac{\partial}{\partial t}\int_{z_G}^{z_{500}}dz=-g\frac{\partial\left(\Delta z\right)}{\partial t}=\int_{p_{SFC}}^{500\text{mb}}\left(\vec{V}\cdot\nabla_p\left(g\frac{\partial z}{\partial p}\right)+\omega\sigma+\frac{R}{pC_p}Q\right)dp$$

Performing ∇_p^2 on the above equation, substituting into the vorticity equation yields:

$$\frac{\partial(\xi_z+f)_{SFC}}{\partial t}=-\vec{V}_{H_{500}}\cdot\nabla_p(\xi_z+f)_{500}+\frac{g}{f}\nabla_p^2\int_{p_{SFC}}^{500}\vec{V}_H\cdot\nabla_p\left(\frac{\partial z}{\partial p}\right)dp$$

$$+\frac{\nabla_p^2}{f}\int_{p_{SFC}}^{500}\omega\sigma\;dp+\frac{R\nabla_p^2}{fC_p}\int_{p_{SFC}}^{500}\frac{Q}{p}dp$$

$$\frac{\partial(\xi_z + f)_{SFC}}{\partial t} = -\vec{V}_{H_{500}} \cdot \nabla_p(\xi_z + f)_{500} + \frac{g}{f}\nabla_p^2 \int\limits_{p_{SFC}}^{500} \vec{V}_H \cdot \nabla_p \left(\frac{\partial z}{\partial p}\right) dp$$

$$+\frac{\nabla_p^2}{f} \int\limits_{p_{SFC}}^{500} \omega\sigma \, dp + \frac{R\nabla_p^2}{fC_p} \int\limits_{p_{SFC}}^{500} \frac{Q}{p} dp$$

This is the Petterssen development equation for the change of surface absolute vorticity due to:

- $-\vec{V}_{H_{500}} \cdot \nabla_p(\xi_z + f)_{500}$: horizontal vorticity advection at 500 mb.
- $\frac{g}{f}\nabla_p^2 \cdot \int\limits_{p_{SFC}}^{500} \vec{V}_H \cdot \nabla_p \left(\frac{\partial z}{\partial p}\right) dp = -\frac{R}{f}\nabla_p^2 \int\limits_{p_{SFC}}^{500} \frac{\vec{V}_H \cdot \nabla_p}{p} (T)\, dp$: proportional to a pressure-weighted horizontal temperature advection between the surface and 500 mb.
- $\frac{\nabla_p^2}{f} \int\limits_{p_{SFC}}^{500mb} \sigma\omega \, dp$: proportional to vertical motion through the layer.
- $\frac{R\nabla_p^2}{fC_p} \int \frac{Q}{p} dp$: proportional to a pressure-weighted diabatic heating pattern.

The Thermal Wind Equation

- The thermal wind equation provides us with information about cold and warm air advection
- Previously we derived the thermal wind equation in Cartesian coordinates – we will now derive the equation using pressure as a vertical coordinate:

The horizontal components of the geostrophic wind in pressure coordinates can be written as:

$$u_g = -\frac{g}{f}\frac{\partial z}{\partial y} \qquad \text{and} \qquad v_g = \frac{g}{f}\frac{\partial z}{\partial x}$$

which can be written using vector notation:

$$\vec{V}_g = \vec{k} \times \frac{g}{f} \nabla_p z$$

$$\vec{V}_g = \vec{k} \times \frac{g}{f} \nabla_p z$$

Differentiating this expression with respect to pressure:

$$\frac{\partial \vec{V}_g}{\partial p} = \frac{g}{f} \vec{k} \times \nabla_p \frac{\partial z}{\partial p}$$

Making use of the hydrostatic asumption and the ideal gas law

which gives $\partial z / \partial p = -1/\rho g = -R_d T / gp$, and substituting this into the

equation above gives:

$$\boxed{\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R_d}{f} \vec{k} \times \vec{\nabla}_p T}$$

the components of which are:

$$\frac{\partial u_g}{\partial \ln p} = \frac{R_d}{f} \frac{\partial T}{\partial y} \quad \text{and} \quad \frac{\partial v_g}{\partial \ln p} = -\frac{R_d}{f} \frac{\partial T}{\partial x}$$

The change of geostrophic wind with pressure is therefore proportional to the

gradient of temperature on a constant pressure surface

$$\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R_d}{f}\vec{k} \times \vec{\nabla}_p T$$

Integrating this equation between two pressure surfaces and rearranging:

$$\int_{\ln p_1}^{\ln p_2} \frac{\partial \vec{V}_g}{\partial \ln p} d\ln p = \vec{V}_{g_{p2}} - \vec{V}_{g_{p1}} = \Delta\vec{V}_g = \frac{R_d}{f}\vec{k} \times \vec{\nabla}_p \overline{T} \ln\left(\frac{p_1}{p_2}\right)$$

where the mean value theorem has been used to take $\vec{\nabla}_p \overline{T}$ from the integral.

$$=> \boxed{\Delta\vec{V}_g = \frac{R_d}{f}\vec{k} \times \vec{\nabla}_p \overline{T} \ln\left(\frac{p_1}{p_2}\right)}$$

The quantity $\Delta\vec{V}_g$ is called the THERMAL WIND

The thermal wind equation can also be written in terms of a thickness gradient.

Performing the gradient operation in the thickness equation $\Delta z = \frac{R_d \overline{T}}{g} \ln(p_1 / p_2)$ gives:

$$\vec{\nabla}_p(\Delta z) = \frac{R_d}{g} \ln\left(\frac{p_1}{p_2}\right) \vec{\nabla}_p \overline{T}$$

which means that the thermal wind equation can be written as:

$$\boxed{\Delta\vec{V}_g = \frac{g}{f}\vec{k} \times \vec{\nabla}_p(\Delta z)}$$

Implications of the Thermal Wind Relations

1) Frontal Strengths

We just saw that :

$$\Delta\vec{V}_g = \frac{g\vec{k}}{f}\times\vec{\nabla}_p(\Delta z)$$

Since the magnitude of $\Delta\vec{V}_g$ is related to the vlaue of the average horizontal temperature gradient through the thickness equation, $\left|\Delta\vec{V}_g\right|$ is used to classify the strength of synoptic fronts. Using the pressure surfaces 1000 mb and 500 mb the following criteria have been established for use by the US Weather Service :

$$\left|\Delta\vec{V}_g\right| < 12.5\text{ m s}^{-1} \quad \rightarrow \quad \text{no front}$$

$$12.5\text{ m s}^{-1} < \left|\Delta\vec{V}_g\right| \leq 25.0\text{ m s}^{-1} \quad \rightarrow \quad \text{weak front}$$

$$25.0\text{ m s}^{-1} < \left|\Delta\vec{V}_g\right| \leq 37.5\text{ m s}^{-1} \quad \rightarrow \quad \text{moderate front}$$

$$37.5\text{ m s}^{-1} < \left|\Delta\vec{V}_g\right| \quad \rightarrow \quad \text{strong front}$$

Since $\overline{V}_g$ at 1000 mb is usually small, strong winds at 500 mb are indicative of a strong front

2) Jet Stream Location

We also saw that:

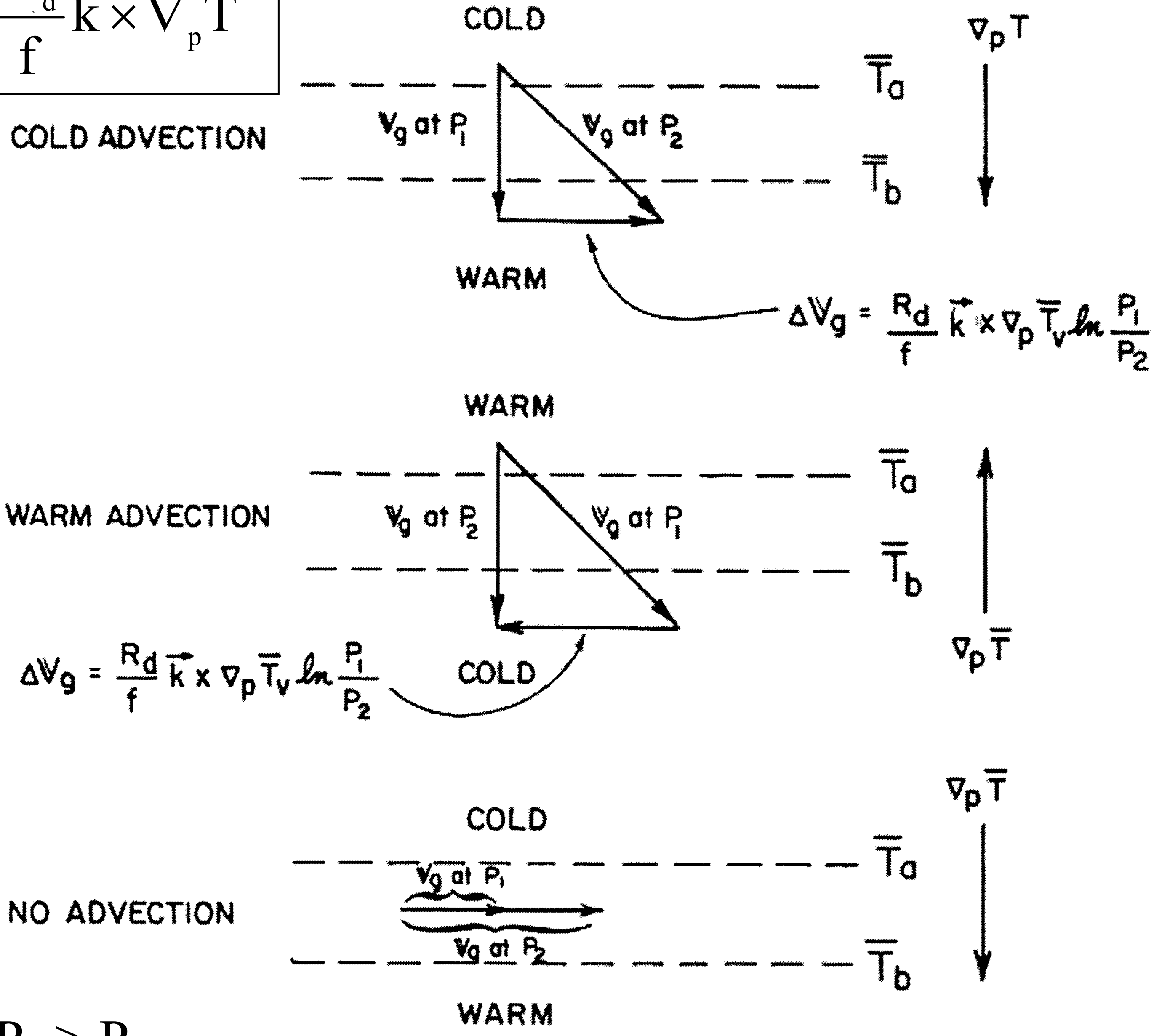
$$\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R_d}{f} \vec{k} \times \vec{\nabla}_p T$$

As the sign of the synoptic temperature gradient is usually the same up to the tropopause, the geostrophic wind continues to increase with height. Above the tropopause, the temperature gradient reverses sign so that the geostrophic wind decreases with height. The region of strongest geostrophic wind near the tropopause is called the jet stream.

3) Temperature Advection

- The magnitude and direction of the thermal wind can be used to estimate temperature advection
- The thermal wind blows parallel to the isotherms with the warm air to the right facing downstream in the Northern Hemisphere
- Geostrophic winds which rotate counterclockwise (back) with height are associated with cold advection in the Northern Hemisphere
- Geostrophic winds which rotate clockwise (veer) with height are associated with warm advection in the Northern Hemisphere
- In the Southern Hemisphere the reverse is true
- In the case of no temperature advection, only the speed of the geostrophic wind, not the direction, changes with height. With cold air towards the pole, this requires that the westerlies increase in speed with height with a low-level westerly geostrophic wind. With warm air to the north, the westerlies would decrease with height.

$$\boxed{\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R_d}{f}\vec{k} \times \vec{\nabla}_p T}$$



$T_a < T_b$, $P_1 > P_2$

Potential Vorticity

- Rossby’s (Barotropic) Potential Vorticity
- Ertel’s Potential Vorticity
- Uses of PV

Rossby (Barotropic) Potential Vorticity

- Earliest mention of potential vorticity was by Rossby (1940)
- Used it to explore the character of flow patterns in the atmosphere and how they change
- Using a barotropic model Rossby defined potential vorticity as:

$$\boxed{PV = \frac{\zeta + f}{h}}$$

  where ζ is the relative vorticity, f is the Coriolis parameter and h is the depth of the fluid.
- For adiabatic, frictionless flow this quantity is conserved
- If we stretch the fluid (increase h) then we must increase the absolute vorticity. For zonal flows this means that as we must increase ζ, resulting in increased cyclonic flow.
- PV contains information about mass and flow fields in one variable

Ertel's Potential Vorticity

- A more general definition for potential vorticity was found by Ertel (1942)
- Ertel used a three-dimensional vector form of the equation of motion for frictionless flow, the thermodynamic equation for adiabatic motion and the mass continuity equation and derived the following conservation principle:

$$\boxed{\frac{D}{Dt}\left(\frac{1}{\rho}\zeta_a \cdot \vec{\nabla}\varphi\right) = 0}$$

where ρ is the density, ζ_a is the absolute vorticity vector, and φ is any conservative thermodynamic variable. In meteorology, φ is typically taken to be potential temperature

- Using potential temperature as our thermodynamic variable, we then get:

$$\frac{DP}{Dt} = \frac{D}{Dt}\left(\frac{1}{\rho}\zeta_a \cdot \vec{\nabla}\theta\right) = 0$$

- The conserved quantity

$$\boxed{P = \frac{1}{\rho}\zeta_a \cdot \vec{\nabla}\theta}$$

is called Ertel's potential vorticity

- Note: the only assumptions are for frictionless, adiabatic flow compared to barotropic model assumptions made by Rossby

- Ertel's theory can be extended to include diabatic and frictional effects (see Cotton's notes for derivation if you are interested)

- Units of PV: K kg^{-1} m^2 s^{-1}
- In meteorology, the following definition is typically used:
    - 1 PVU = 10^{-6} K kg^{-1} m^2 s^{-1} where PVU stands for potential vorticity units
    - Values of IPV less than ~1.5 PVU are usually associated with tropospheric air
    - IPV values larger than 1.5 PVU are usually associated with stratospheric air

- Isentropic Potential Vorticity (IPV) is given by:

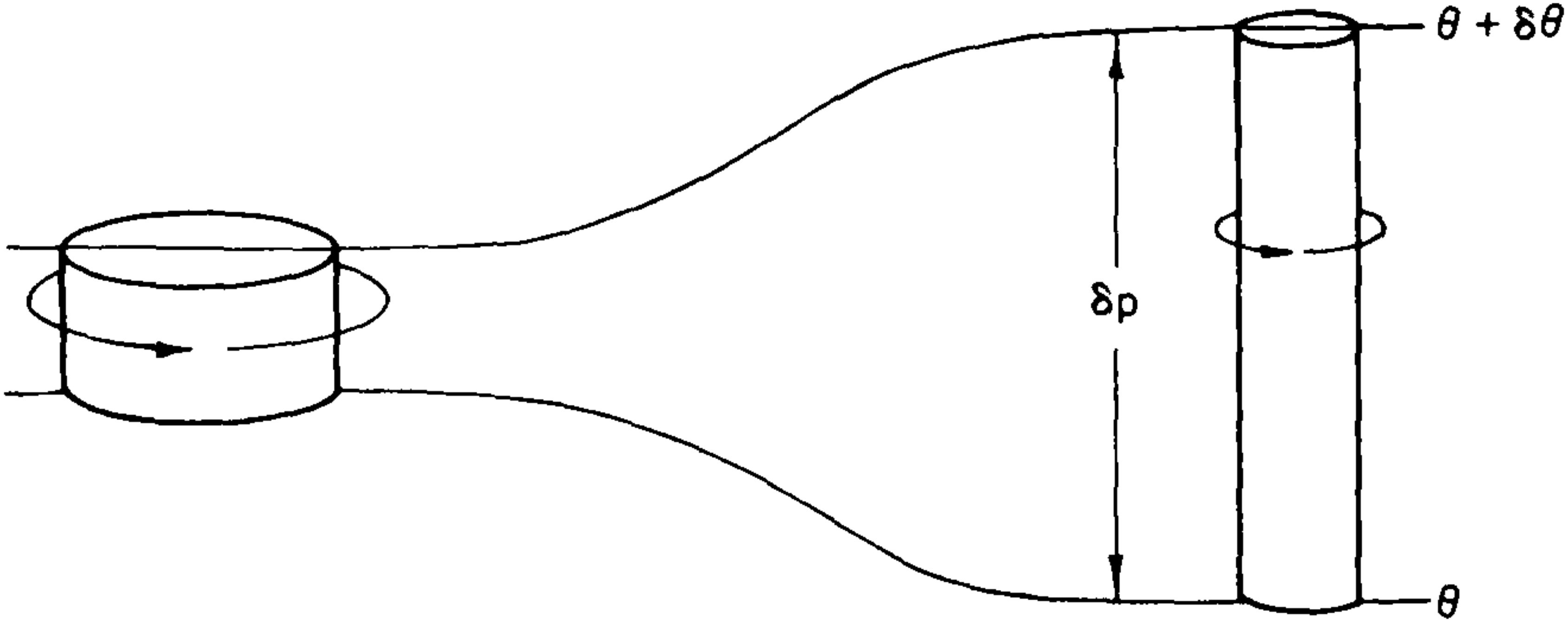
$$\boxed{P = \underbrace{(\zeta_\theta + f)}_{\text{Absolute vorticity}}\underbrace{(-g\frac{\partial\theta}{\partial p})}_{\text{Static stability}} = \text{const}}$$

where ζ_θ is the vertical component of relative vorticity evaluated on an isentropic surface

- Potential vorticity is therefore the product of the absolute vorticity and the static stability. If the static stability is increased (i.e., if $\partial\theta/\partial p$ is made more negative), absolute vorticity (which is positive) is decreased and vice versa.
- The "potential" in potential vorticity relates to the value the relative vorticity would have if a parcel is moved adiabatically to a standard latitude and static stability.

- IPV is defined with a minus sign so that its value is normally positive in the Northern Hemisphere
- The expression on the previous slide shows that potential vorticity is conserved following the motion in adiabatic, frictionless flow
- PV is always in some sense a measure of the ratio of the absolute vorticity to the effective depth of the vortex. In the expression on the previous slide, the effective depth is just the distance between isentropic surfaces measured in pressure units ($-\partial\theta/\partial p$)
- The conservation of PV is a powerful constraint on large-scale motions of the atmosphere

$$P = \left(\zeta_{\theta} + f\right)\left(-g\frac{\partial\theta}{\partial p}\right) = const$$



A cylindrical column of air moving adiabatically, conserving potential vorticity

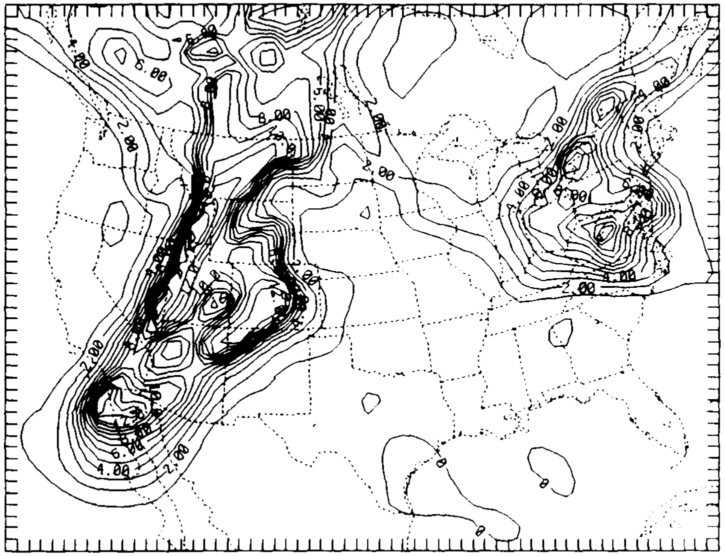
Ref: Holton

Uses of PV

1. Surface cyclones have been found to be accompanied by a positive PV anomaly (high PV air relative to the environment) aloft
2. Tracer of stratospheric air
   - As PV is a function of static stability, regions with strong static stability should also be regions of high PV
   - The stratosphere has been recognized as a possible source region or reservoir of high-PV air – above the tropopause θ rapidly increases with height and hence so does PV
   - PV values larger than 1.5 PVU are usually associated with stratospheric air

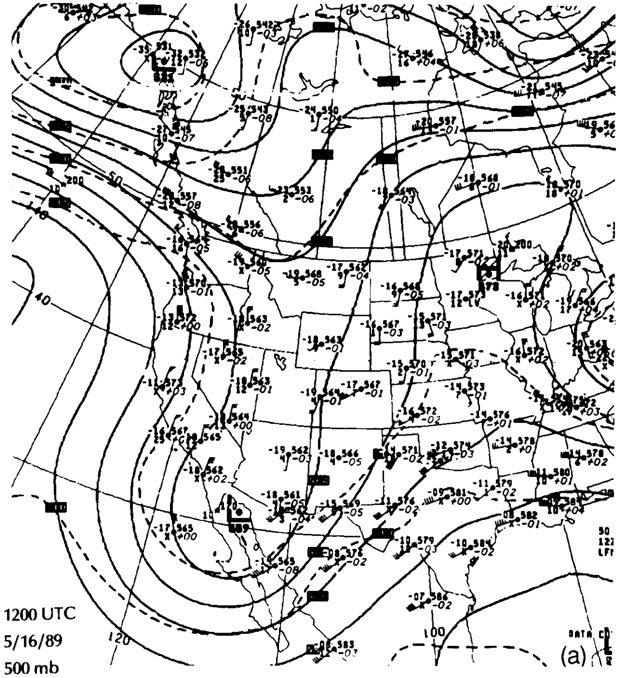
3. Indicator or troughs and ridges, as well as closed lows and highs

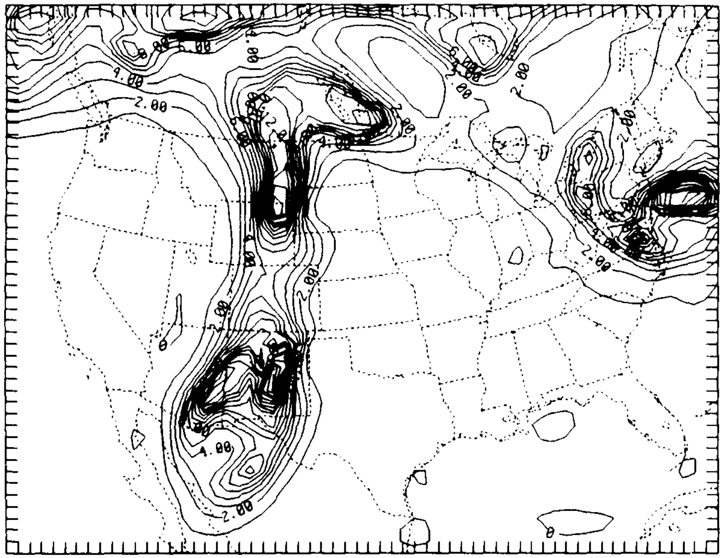
- Tongues of high-IPV(low-IPV), stratospheric (tropospheric) air that extend equatorward (poleward) from the high-IPV reservoir (low-IPV troposphere) are associated with troughs (ridges) in the height field and cyclonic (anticyclonic) flow
- Isolated regions of stratospheric IPV that are situated equatorward from the reservoir tend to be associated with troughs or closed lows in the height field and associated cyclonic flow



1200 UTC 5/16/89

(a)





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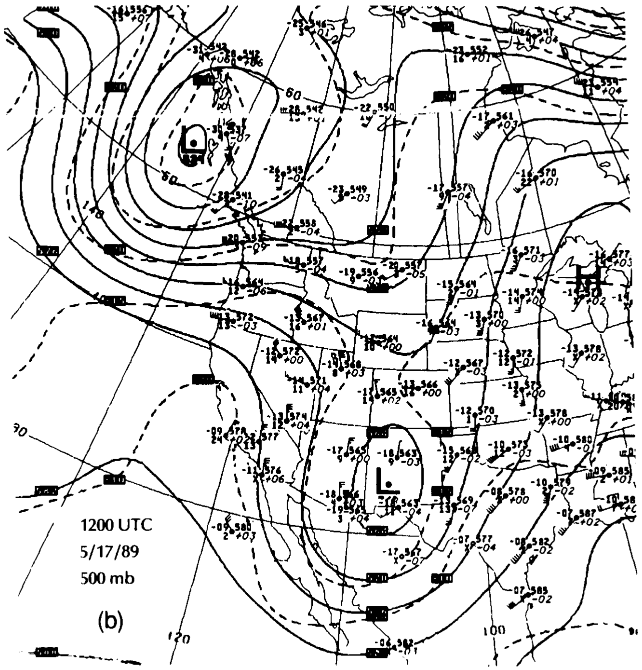
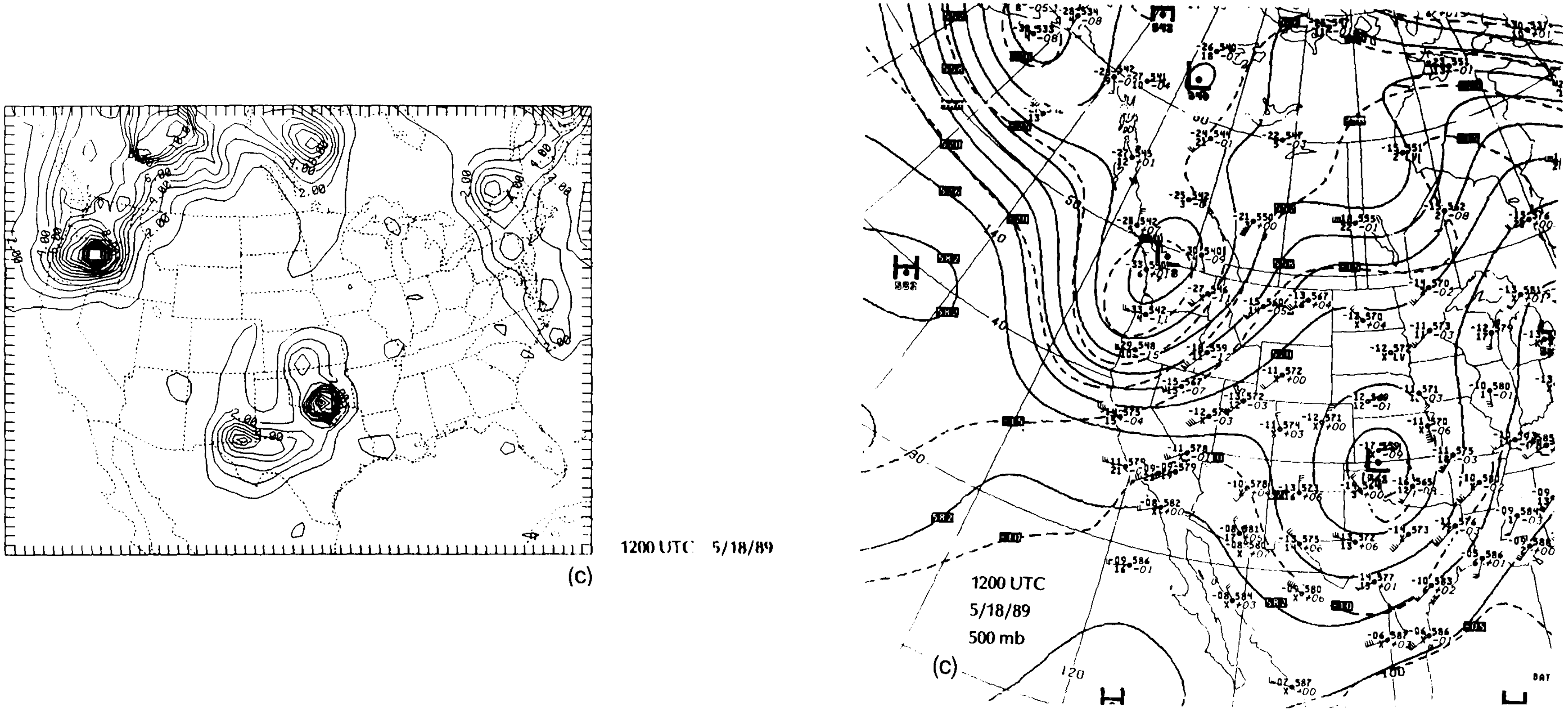




Ertel’s PV (PVU) evaluated on the 325K surface (left) on (a) 1200 UTC, May 16, (b) 1200 UTC, May 17, and (c) 1200 UTC, May 18, 1989. Corresponding 500 mb height contours, temperature and dew-point depression (°C). Ref: Bluestein

4. Lee Cyclogenesis

- Westerly Flow
  - Suppose that upstream of the mountain barrier flow is a uniform zonal flow => $\zeta=0$
  - If the flow is adiabatic, each column of air is confined between the potential temperature surfaces θ_0 and $\theta_0 + \delta\theta$ as it crosses the mountain
  - Potential temperature surface θ_0 near the ground approximately follows the contours of the ground. A potential temperature surface $\theta_0 + \delta\theta$ several kilometers above the ground will also be deflected vertically, however, the vertical displacement at upper levels is spread horizontally and has less displacement in the vertical than that near the ground

- Due to the vertical displacement of the upper-level isentropic surfaces there is a vertical stretching of air columns upstream of the topographic barrier

  => causes $-\partial\theta/\partial p$ to decrease

  => ζ must become positive to conserve PV

  => air column turns cyclonically as it approaches the topographic barrier

  => the cyclonic curvature causes a poleward drift so that f also increases which reduces the change in ζ required for PV conservation

- As the column begins to cross the barrier its vertical extent decreases

  => relative vorticity must then become negative

  => air column will acquire anticyclonic vorticity and move southward

  - Once the air column has passed over the mountain and returned to its original depth it will be south of its original latitude so that f will be smaller and the relative vorticity must be positive

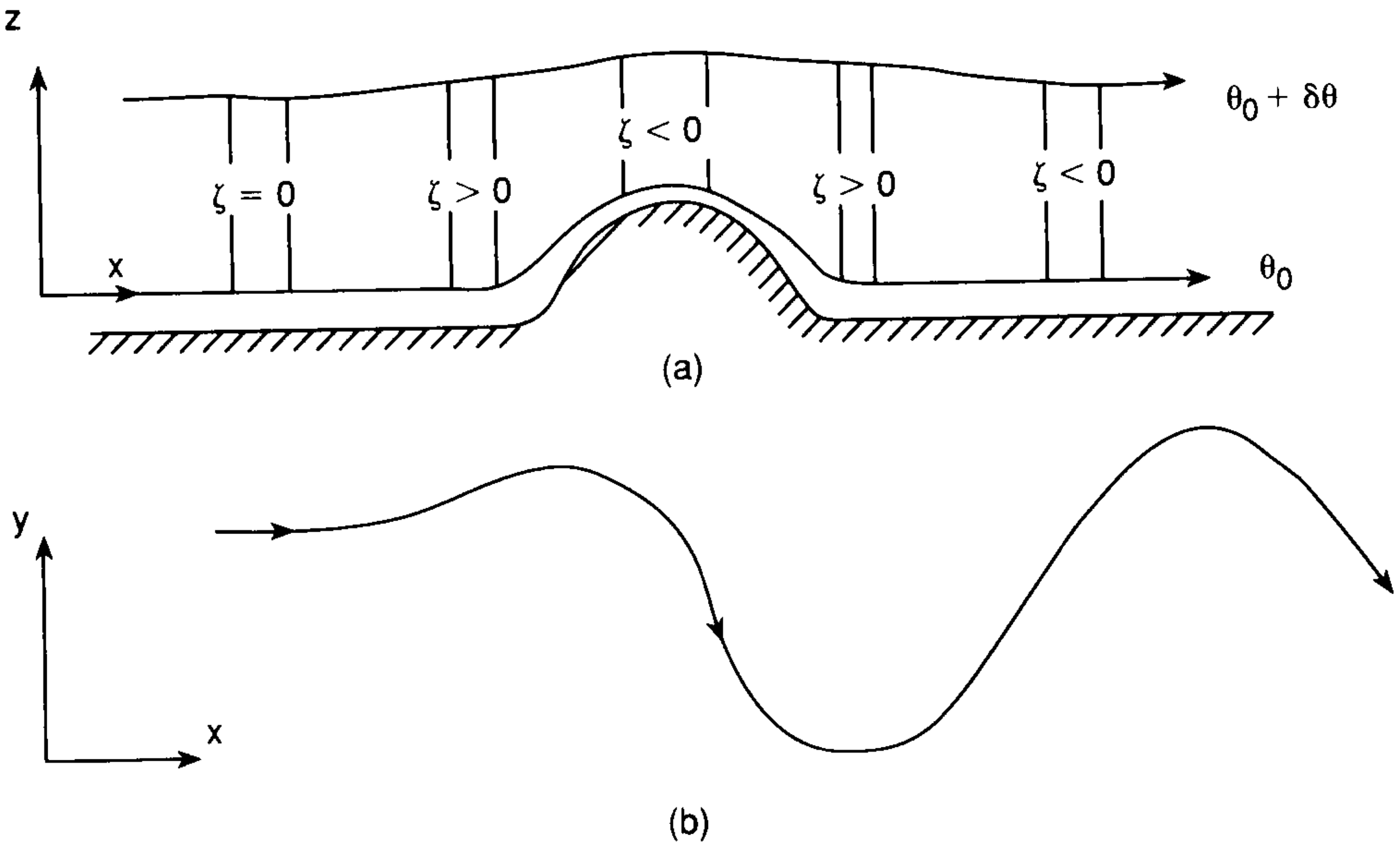
    => trajectory must have cyclonic curvature

    => column will be deflected poleward

    => when parcel reaches its original latitude it will still have a poleward velocity component and will continue poleward gradually acquiring anticyclonic curvature until its direction is reversed

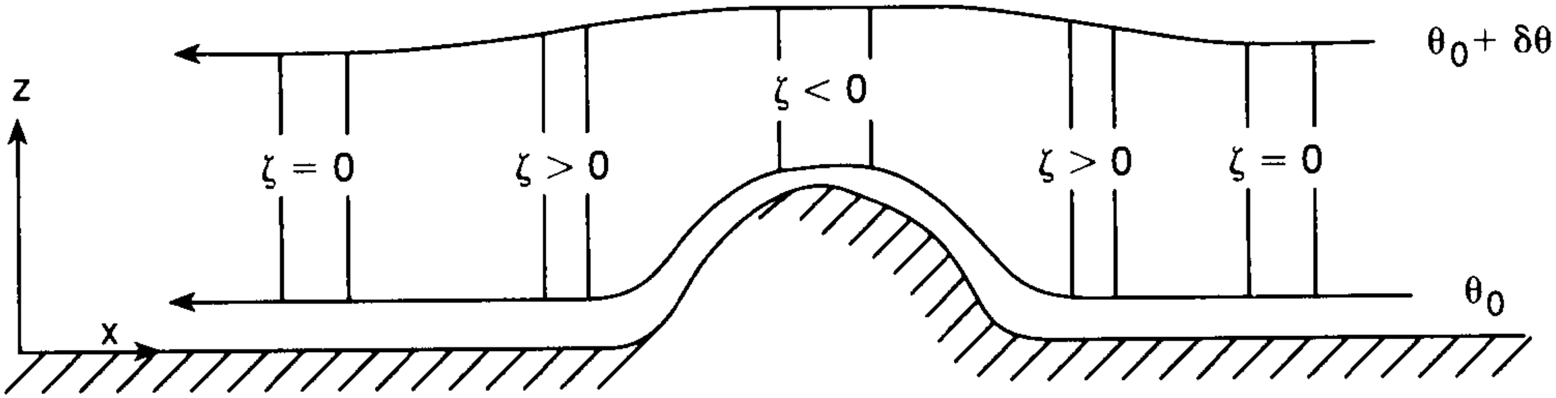
    => parcel will then move downstream conserving PV by following a wave-like trajectory in the horizontal plane

- A steady westerly flow over a large-scale mountain barrier will therefore result in a cyclonic flow pattern immediately to the east of the barrier (called the lee side trough) followed by an alternating series of ridges and troughs

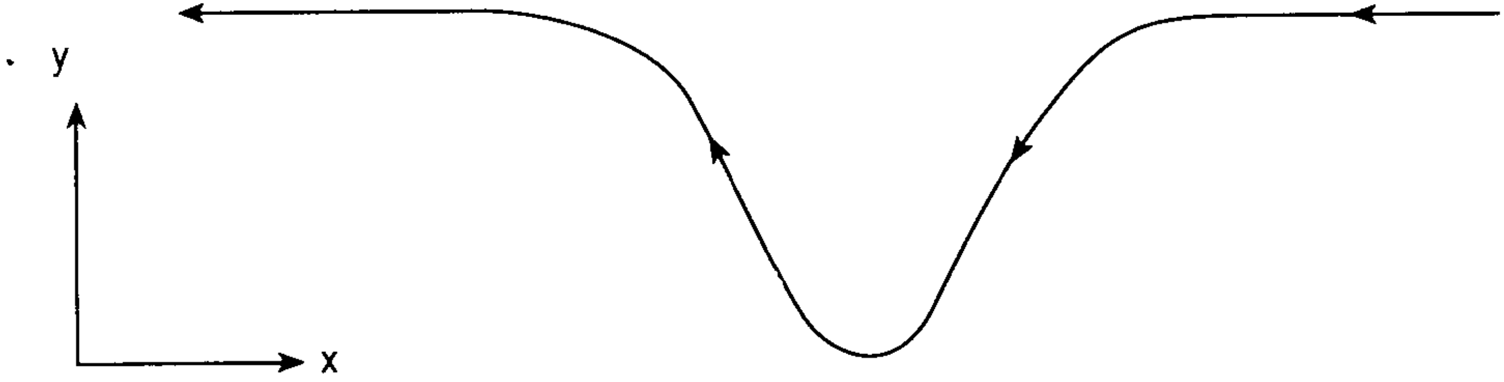


Schematic view of westerly flow over a topographic barrier: (a) the depth of the fluid column as a function of x and (b) the trajectory of a parcel in the x,y plane Ref: Holton

- Easterly Flow
  - Similar reasoning may be applied to obtain the effects of a large topographic barrier on a purely zonal easterly flow
  - There is a dramatic difference between easterly and westerly flow over large-scale topographic barriers
  - In the westerly wind case the barrier generates wavelike disturbances in the streamlines that extend downwind from the barrier
  - In the easterly wind case the disturbance in the streamlines damps out away from the barrier
  - The differences are due to the dependence of the Coriolis parameter on latitude



(a)



(b)

Schematic view of easterly flow over a topographic barrier: (a) the depth of the fluid column as a function of x and (b) the trajectory of a parcel in the x,y plane Ref: Holton

Midterm Exam: General Notes

- Mid-term exam: Thursday October 16, 12:10 – 2:10
- Includes theory and lab applications, weighted more heavily toward theory
- Derivations are fair game although the following derivations will NOT be included: virtual temperature, enthalpy, vorticity equation, Omega equation, Q vector form of the Omega equation, and Petterssen's equation.
- You must however understand the final form of the various equations, know what basic equations and assumptions are used in the derivations, be able to interpret their terms and use these equations in explanations of various weather phenomena
- Bring colored pencils
- No programmable calculators

Chapter 2 Outline

- The primary variables
- Instrumentation used to measure the primary variables
- Remote sensing
  - Electromagnetic spectrum and associated laws
  - Instrumentation: satellite, radar, wind profilers

Chapter 3 Outline

- The Gas Laws
  - Ideal gas law: general, dry air, water vapor
  - Universal Gas constant, gas constant for dry air and water vapor
  - Virtual temperature
- Hydrostatic Equation
- Geopotential and geopotential height
  - High and low pressure, ridges and troughs
- Thickness
  - Thickness / Hypsometric equation
  - Warm and cold core systems
  - Uses of thickness: frontal location, rain-snow line, warm or cold air advection
- First Law of Thermodynamics
  - Various forms of the first law
  - Joule’s law
- Specific Heats
  - At constant volume
  - At constant pressure

- Enthaply
- Potential temperature
- Dry Adiabatic Lapse Rate
- Water vapor and moisture parameters:
    - Mixing ratio
    - Specific humidity
    - Saturation vapor pressure with respect to liquid water and ice
    - Saturation mixing ratios
    - Dew point temperature and frost point temperature
    - Lifting condensation level
    - Wet bulb temperature
- Saturated Adiabatic Lapse Rate
- Equivalent potential temperature
- Static stability
- Conditional instability
- Convective or potential instability

Chapter 4 Outline

- Coordinate Systems
  - Velocity Components
  - Pressure as a vertical coordinate
  - Other vertical coordinates
  - Natural coordinates
- Apparent Forces
  - Coriolis Force
  - Effective gravity
- Thermal Wind
- Balance Winds
  - Geostrophic, gradient, cyclostrophic, friction

- Continuity Equation
  - Material derivative form, flux form,
  - Incompressibility, convergence, divergence, level of nondivergence, vertical motion
  - Pressure tendency equation
- Baroclinity and barotropy
- Vorticity
  - Components
  - Absolute and relative vorticity
  - Vorticity equation and interpretation of all terms
  - Rotational and shear vorticity
- Omega Equation
- Q vectors
- Petterssen’s Developmental Equation
- Potential Vorticity