

Course Outline

	Section	Instructor	Time
1	Atmospheric Variables	Cotton	1 class
2	Communication with the atmos	Jirak	1 class
3	Atmospheric Thermodynamics	vd Heever	~4 weeks
4	Atmospheric Dynamics	vd Heever	~4 weeks
5	Extratropical Cyclones	Jirak	~2 weeks
6	Numerical Weather Prediction	vd Heever	~1 week
7	Conceptual Models of Cumulonimbi	Cotton	1-2 classes
8	Conceptual Models of MCSs	Cotton	1-2 classes
9	Tropical Cyclones	Cotton	1-2 classes

Focus

- Practical application of basic meteorological principles and laws
- Brings together aspects of many classes – dynamics, thermodynamics, synoptic meteorology
- Forecasting techniques and ability to interpret weather maps
- Introduce new terminology for students not having previously taken any meteorology / atmospheric science classes
- Some derivations
- Notes based on Wallace and Hobbs, Cotton's notes, Pielke's notes and my own notes

3. Atmospheric Thermodynamics

1. The gas laws
2. The hydrostatic equation
3. Definition of geopotential and geopotential height
4. Thickness
5. First law of thermodynamics
6. Specific heats
7. Enthalpy
8. Potential temperature
9. The adiabatic lapse rate
10. Water vapor and moisture parameters
11. Saturated-adiabatic and pseudo adiabatic processes
12. Equivalent potential temperature
13. Static Stability

3.1 The Gas Laws

- Lab experiments: pressure, volume and temperature of any material can be related by an **Equation of State**
- All gases follow approximately the same equation of state – **Ideal Gas Equation**
- Ideal gas equation or Equation of State for gases:

$$pV = mRT$$

where p is pressure (Pa)

V is volume (m³)

m is mass (kg)

T is temperature (K)

R is the gas constant (J K⁻¹ kg⁻¹)

- R is dependant on the gas type

$$pV = mRT$$

$$\text{Now } \frac{m}{V} = \rho$$

ρ is the density of the gas

$$\therefore p = \rho RT$$

For unit mass (1 kg), $m = 1$ and

$$\alpha = \frac{1}{\rho}$$

α is the specific volume

$$\therefore p\alpha = RT$$

Atomic weight of
hydrogen 1.00794 AMU

Periodic Table of the Elements

Atomic weight of oxygen 15.9944 AMU

[illegible]

Atomic masses in parentheses are those of the most stable or common isotope.

Concepts and Definitions

- Atomic weight and atomic mass are used interchangeably, as are molecular weight and molecular mass
- On the periodic table, atomic masses are **relative** and are given in atomic mass units (AMU)
- Examples:
 - Hydrogen: 1.0079
 - Oxygen: 15.9994
 - Water: 18.0153

Concepts and Definitions (cont)

- A **gram-molecular weight or mole** (mol) of any substance is the molecular weight of the substance expressed in grams
- A mole of a gas therefore contains enough molecules to give the molecular mass in grams
- Examples:
 - 1 mol of hydrogen weighs 1.0079 g
 - 1 mol of oxygen weighs 15.9994 g
 - 1 mol of water weighs 18.0153 g

- As molecular masses are relative, 1 mol of any substance must contain the same number of molecules as 1 mol of any other substance.
- The number of molecules in 1 mol of any substance is a universal constant, which is called **Avagadro's number**, N_A . The value of N_A is 6.022×10^{23} per mole.
- The number of moles in the mass of a substance is given by:

$$n = \frac{m}{M}$$

n : number of moles

m : mass (g)

M : molecular mass (g/mol)

- Example: For 18 g of hydrogen we have ~18 mol of hydrogen, whereas for 18g of water we have ~1 mol of water
- **Avagadro's hypothesis**: Gases containing the same number of molecules occupy the same volumes at the same temperature and pressure.

Universal Gas Constant

Assume that we have x molecules of gas A and x molecules of gas B at some temperature T and some pressure P . We can then write the equation of state for x molecules of each gas as:

$$pV_A = R_A T$$

$$pV_B = R_B T$$

But from Avagadro's hypothesis, gases containing the same number of molecules occupy the same volumes at the same temperature and pressure:

$$\Rightarrow V_A = V_B$$

$$\therefore R_A = R_B$$

So for the same number of molecules of any gas, the constant R will be the same!

- As 1 mol of gas contains the same number of molecules as 1 mol of any other gas $\rightarrow$ R for 1 mol is the same for all gases
- Called the **Universal Gas Constant (R^*)**
- $R^* = 8.3145 \text{ J K}^{-1} \text{ mol}^{-1}$
- The ideal gas law for 1 mol of any gas:

$$pV = R^*T$$

and for n moles of any gas is:

$$pV = nR^*T$$

So for some mass of gas we have: $pV = mRT$

where R is the gas constant for that particular gas ($\text{JK}^{-1}\text{kg}^{-1}$)

and for some number of moles we have: $pV = nR^*T$

where R^* is the Universal Gas Constant ($8.3145 \text{ JK}^{-1}\text{mol}^{-1}$)

$$\text{Now } n = \frac{m}{M} \Rightarrow m = nM$$

$$\therefore pV = nMRT$$

$$\Rightarrow MR = R^*$$

$$\Rightarrow R = \frac{R^*}{M}$$

For a gaseous mixture the molecular mass is: $M = \frac{\sum_i m_i}{\sum_i \frac{m_i}{M_i}}$
where m_i and M_i are the mass (g) and
molecular mass of the i^{th} constituent

Dry Air

Equation of state for unit mass :

$$p_d \alpha_d = R_d T \quad \text{or} \quad p_d = \rho_d R_d T$$

where R_d is the gas constant for dry air.

$$M_d = 28.97$$

$\therefore$ 1 mol of dry air has a $M_d = 28.97 \text{ g}$

$$\Rightarrow R_d = 1000 \frac{R^*}{M_d} = 1000 \frac{R^*}{28.97} = 287 \text{ J K}^{-1} \text{ kg}^{-1}$$

Water Vapor

Equation of state for unit mass :

$$e\alpha_v = R_v T \quad \text{or} \quad e = \rho_v R_v T$$

where R_v is the gas constant for water vapor
and e is vapor pressure

$$M_v = 18.016$$

$\therefore$ 1 mol of water vapor has a $M_v = 18.016\text{g}$

$$\Rightarrow R_v = 1000 \frac{R^*}{M_v} = 1000 \frac{R^*}{18.016} = 461 \text{ J K}^{-1} \text{ kg}^{-1}$$

Gas Constants

- So we have:

$$R_d = 287 \text{ J K}^{-1} \text{ kg}^{-1}$$

$$R_v = 461 \text{ J K}^{-1} \text{ kg}^{-1}$$

$$R^* = 8.3145 \text{ JK}^{-1}\text{mol}^{-1}$$

- Define:

$$\varepsilon = \frac{R_d}{R_v} = \frac{M_v}{M_d} = 0.622$$

Dalton's Law of Partial Pressures

- The total pressure exerted by a mixture of gases that do not interact chemically is equal to the sum of the partial pressures of the gases
- The partial pressure of a gas is the pressure that that gas would exert if it alone occupied the volume that the mixture occupies (at the same temperature)

$$p = \sum p_i$$

For moist air :

$$p = \sum p_i = p_d + e$$

$$\rho = \frac{m_d + m_v}{V} = \frac{m_d}{V} + \frac{m_v}{V} = \rho'_d + \rho'_v \quad \text{(partial densities)}$$

Example (Wallace and Hobbs)

- If at 0°C the density of dry air alone is 1.275 kg m^{-3} and, under the same conditions, the density of water vapor alone is $4.770 \times 10^{-3} \text{ kg m}^{-3}$, what is the total pressure exerted by a mixture of dry air and water vapor at 0°C?

$$p_d = \rho_d R_d T$$

$$\text{where } \rho_d = 1.275 \text{ kg m}^{-3}, R_d = 287 \text{ J K}^{-1} \text{ kg}^{-1}, T = 273 \text{ K}$$

$$\Rightarrow p_d = 9.99 \times 10^4 \text{ Pa} = 999 \text{ hPa}$$

$$e = \rho_v R_v T$$

$$\text{where } \rho_v = 4.770 \times 10^{-3} \text{ kg m}^{-3}, R_v = 461 \text{ J K}^{-1} \text{ kg}^{-1}, T = 273 \text{ K}$$

$$\Rightarrow e = 600 \text{ Pa} = 6 \text{ hPa}$$

$\therefore$ the total pressure exerted by the mixture of dry air and water vapor is $(999 + 6) \text{ hPa}$ which is 1005 hPa .

Virtual Temperature

- Moist air has a smaller molecular mass than dry air

$$\Rightarrow R_d = R^* / M_d < R_m = R^* / M_m$$

- Value of R_m varies depending on amount of water vapor in the air
- More convenient to use R_d – how can we do this?

Density of moist air :

$$\rho = \frac{m_d + m_v}{V} = \rho'_d + \rho'_v \quad (\text{partial densities})$$

$$e = \rho'_v R_v T \quad \text{and} \quad p_d = \rho'_d R_d T$$

$$\rho = \rho'_d + \rho'_v = \frac{p_d}{R_d T} + \frac{e}{R_v T} = \frac{p_d}{R_d T} + \frac{e}{R_d \left(\frac{R_v}{R_d} \right) T}$$

$$\rho = \frac{p - e}{R_d T} + \frac{e}{R_d \frac{1}{\varepsilon} T} = \frac{p}{R_d T} \left(1 - \frac{e}{p} + \frac{e\varepsilon}{p} \right) \quad (\text{remember } \varepsilon = R_d / R_v)$$

$$= \frac{p}{R_d T} \left(1 - \frac{e}{p} (1 - \varepsilon) \right)$$

$$\text{Rearranging: } p = \rho R_d T \left(\frac{1}{1 - \frac{e}{p} (1 - \varepsilon)} \right)$$

$$\text{Define } T_v = T / \left(1 - \frac{e}{p} (1 - \varepsilon) \right)$$

$$\Rightarrow p = \rho R_d T_v$$

$$p = \rho R_d T_v \quad \text{where } T_v = T / \left(1 - \frac{e}{p} (1 - \varepsilon) \right)$$

- T_v is called the virtual temperature
- T_v is a fictitious temperature – cannot actually measure it!
- T_v is the temperature that dry would have to be to have the same density as moist air at the same pressure
- As moist air is always less dense than dry air: $T_v > T$
- Adding moisture to the air increases T_v
- T_v is actually only a few degrees greater than T even for very warm, moist air

Equation of State: Summary

- $pV=mRT$
- $p\alpha=RT$
- $p=\rho RT$
- $pV=nR^*T,$ $R^* = 8.3145 \text{ J K}^{-1}\text{mol}^{-1}$
- Dry air: $p_d=\rho_d R_d T,$ $R_d = 287 \text{ J K}^{-1} \text{ kg}^{-1}$
- Water vapor: $e=\rho_v R_v T,$ $R_v = 461 \text{ J K}^{-1} \text{ kg}^{-1}$
- $p=\rho R_d T_v$

3.2 Hydrostatic Equation

- Air pressure at some level in the atmosphere is due to the weight of the air in the column above this level

=> air pressure decreases with height

- **Hydrostatic Balance**: when the net upward force on a thin horizontal slab of air is equal to the downward force on this slab the atmosphere is said to be in hydrostatic balance

- Consider a vertical column of air between z and $z + \delta z$ with a unit cross-sectional area

Mass of slab between z and $z + \delta z$:

$$\rho = \frac{m}{V} \text{ and } V = \text{area} \times \text{height} = \delta z$$

$$\Rightarrow m = \rho V = \rho \delta z$$

Downward force acting on slab:

- weight of the slab
- $F = m \times a = \rho \delta z \times g = g \rho \delta z$

Upward force acting on slab:

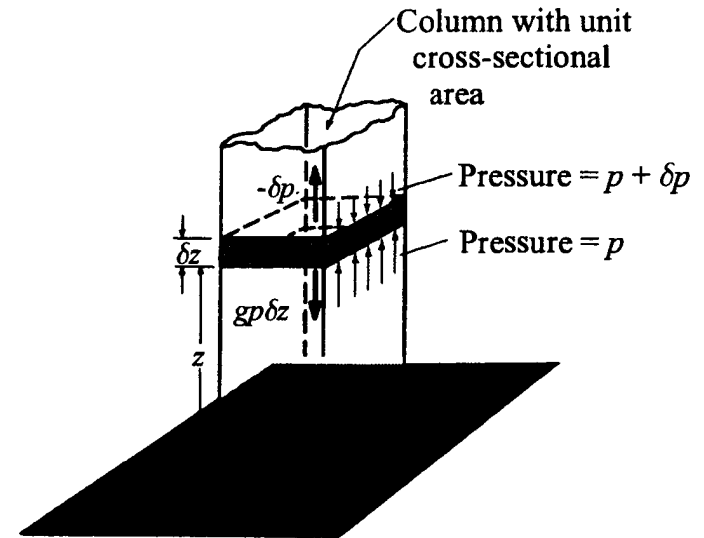
- force due to the vertical pressure gradient
- $p = \text{force} / \text{area} = \text{force}$ as we are considering unit area
- δp represents the force due to the pressure gradient
- $-\delta p$ acts upward as pressure decreases with height

For **Hydrostatic Balance**:

$$-\delta p = g \rho \delta z$$

$$\Rightarrow \frac{dp}{dz} = -\rho g$$

Hydrostatic Equation



An Alternate View

Vertical Equation of Motion :

$$\frac{dw}{dt} = -\frac{1}{\rho} \frac{dp}{dz} - g + F$$

where $\frac{dw}{dt}$ is the vertical acceleration

$-\frac{1}{\rho} \frac{dp}{dz}$ is the pressure gradient force

g is the acceleration due to gravity

F is the friction term

If we are to have hydrostatic balance (ignoring friction) then $\frac{dw}{dt} = 0$

$$\Rightarrow -\frac{1}{\rho} \frac{dp}{dz} = g$$

$$\Rightarrow \frac{dp}{dz} = -\rho g$$

Hydrostatic Equation Notes

- $\frac{dp}{dz} = -\rho g$ should really be written as: $\frac{\partial p}{\partial z} = -\rho g$
- The negative sign ensures that pressure decreases with height
- Given that $\rho = 1/\alpha$ we can write

$$g dz = -\alpha dp$$

- To find the pressure at any height z :

$$-\int_{p(z)}^{p(\infty)=0} dp = \int_z^{\infty} g \rho dz \quad \Rightarrow$$

$$p(z) = \int_z^{\infty} g \rho dz$$

- **1 atmosphere** (or 1 atm): the pressure at sea level is 1013 hPa (if the mass of the atmosphere is uniformly distributed over the earth)

3.3 Geopotential and Geopotential Height

- Definition: the geopotential (ϕ) is the work that must be done against the gravitational field to raise a mass of 1 kg from sea level to a given height
- The geopotential is the potential energy of a unit mass relative to sea level
- Units: work per unit mass (J kg⁻¹ or m²s⁻²)
- Work = force x distance
= m x a x distance
= g x distance (for unit mass)
- The work needed to raise 1kg from z to z+ δZ is therefore gdz
- The geopotential at some height z is therefore:
 $\Rightarrow d\phi = gdz = -\alpha dp$ (using the hydrostatic equation)

$$\phi(z) = \int_{\phi(0)=0}^{\phi(z)} d\phi = \int_0^z gdz$$

- Define the geopotential height:

$$Z = \frac{\phi(z)}{g_0} = \frac{1}{g_0} \int_0^z g dz$$

where g_0 is the globally averaged acceleration due to gravity at the Earth's surface

- $g_0 = 9.81 \text{ m.s}^{-2}$
- In the lower atmosphere, $g \approx g_0$
- Z is plotted on our weather maps

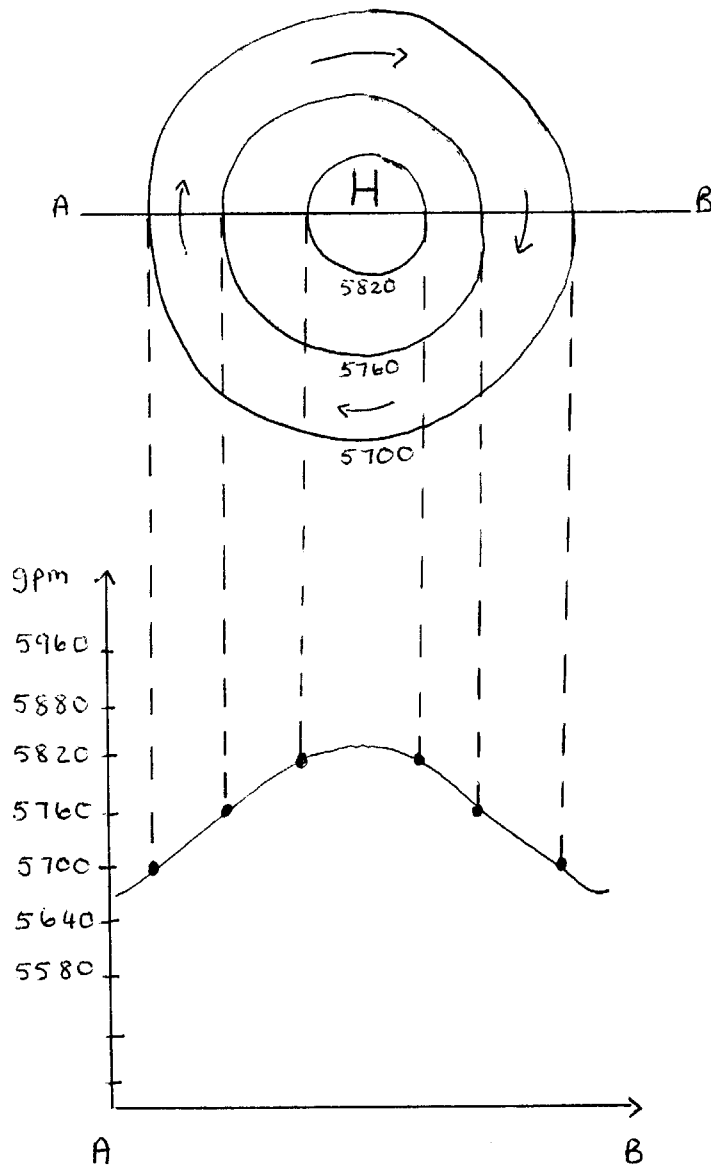
500 height temp for 12Z 15 AUG 05

1013 1008 1004 1000 996 992 988 984 980 976 972 968 964 960 956 952 948 944 940 936 932 928 924 920 916 912 908 904 900 896 892 888 884 880 876 872 868 864 860 856 852 848 844 840 836 832 828 824 820 816 812 808 804 800 796 792 788 784 780 776 772 768 764 760 756 752 748 744 740 736 732 728 724 720 716 712 708 704 700 696 692 688 684 680 676 672 668 664 660 656 652 648 644 640 636 632 628 624 620 616 612 608 604 600 596 592 588 584 580 576 572 568 564 560 556 552 548 544 540 536 532 528 524 520 516 512 508 504 500 496 492 488 484 480 476 472 468 464 460 456 452 448 444 440 436 432 428 424 420 416 412 408 404 400 396 392 388 384 380 376 372 368 364 360 356 352 348 344 340 336 332 328 324 320 316 312 308 304 300 296 292 288 284 280 276 272 268 264 260 256 252 248 244 240 236 232 228 224 220 216 212 208 204 200 196 192 188 184 180 176 172 168 164 160 156 152 148 144 140 136 132 128 124 120 116 112 108 104 100 96 92 88 84 80 76 72 68 64 60 56 52 48 44 40 36 32 28 24 20 16 12 8 4 0 -4 -8 -12 -16 -20 -24 -28 -32 -36 -40 -44 -48 -52 -56 -60 -64 -68 -72 -76 -80 -84 -88 -92 -96 -100 -104 -108 -112 -116 -120 -124 -128 -132 -136 -140 -144 -148 -152 -156 -160 -164 -168 -172 -176 -180 -184 -188 -192 -196 -200 -204 -208 -212 -216 -220 -224 -228 -232 -236 -240 -244 -248 -252 -256 -260 -264 -268 -272 -276 -280 -284 -288 -292 -296 -300 -304 -308 -312 -316 -320 -324 -328 -332 -336 -340 -344 -348 -352 -356 -360 -364 -368 -372 -376 -380 -384 -388 -392 -396 -400 -404 -408 -412 -416 -420 -424 -428 -432 -436 -440 -444 -448 -452 -456 -460 -464 -468 -472 -476 -480 -484 -488 -492 -496 -500 -504 -508 -512 -516 -520 -524 -528 -532 -536 -540 -544 -548 -552 -556 -560 -564 -568 -572 -576 -580 -584 -588 -592 -596 -600 -604 -608 -612 -616 -620 -624 -628 -632 -636 -640 -644 -648 -652 -656 -660 -664 -668 -672 -676 -680 -684 -688 -692 -696 -700 -704 -708 -712 -716 -720 -724 -728 -732 -736 -740 -744 -748 -752 -756 -760 -764 -768 -772 -776 -780 -784 -788 -792 -796 -800 -804 -808 -812 -816 -820 -824 -828 -832 -836 -840 -844 -848 -852 -856 -860 -864 -868 -872 -876 -880 -884 -888 -892 -896 -900 -904 -908 -912 -916 -920 -924 -928 -932 -936 -940 -944 -948 -952 -956 -960 -964 -968 -972 -976 -980 -984 -988 -992 -996 -1000 -1004 -1008 -1012 -1016 -1020 -1024 -1028 -1032 -1036 -1040 -1044 -1048 -1052 -1056 -1060 -1064 -1068 -1072 -1076 -1080 -1084 -1088 -1092 -1096 -1100 -1104 -1108 -1112 -1116 -1120 -1124 -1128 -1132 -1136 -1140 -1144 -1148 -1152 -1156 -1160 -1164 -1168 -1172 -1176 -1180 -1184 -1188 -1192 -1196 -1200 -1204 -1208 -1212 -1216 -1220 -1224 -1228 -1232 -1236 -1240 -1244 -1248 -1252 -1256 -1260 -1264 -1268 -1272 -1276 -1280 -1284 -1288 -1292 -1296 -1300 -1304 -1308 -1312 -1316 -1320 -1324 -1328 -1332 -1336 -1340 -1344 -1348 -1352 -1356 -1360 -1364 -1368 -1372 -1376 -1380 -1384 -1388 -1392 -1396 -1400 -1404 -1408 -1412 -1416 -1420 -1424 -1428 -1432 -1436 -1440 -1444 -1448 -1452 -1456 -1460 -1464 -1468 -1472 -1476 -1480 -1484 -1488 -1492 -1496 -1500 -1504 -1508 -1512 -1516 -1520 -1524 -1528 -1532 -1536 -1540 -1544 -1548 -1552 -1556 -1560 -1564 -1568 -1572 -1576 -1580 -1584 -1588 -1592 -1596 -1600 -1604 -1608 -1612 -1616 -1620 -1624 -1628 -1632 -1636 -1640 -1644 -1648 -1652 -1656 -1660 -1664 -1668 -1672 -1676 -1680 -1684 -1688 -1692 -1696 -1700 -1704 -1708 -1712 -1716 -1720 -1724 -1728 -1732 -1736 -1740 -1744 -1748 -1752 -1756 -1760 -1764 -1768 -1772 -1776 -1780 -1784 -1788 -1792 -1796 -1800 -1804 -1808 -1812 -1816 -1820 -1824 -1828 -1832 -1836 -1840 -1844 -1848 -1852 -1856 -1860 -1864 -1868 -1872 -1876 -1880 -1884 -1888 -1892 -1896 -1900 -1904 -1908 -1912 -1916 -1920 -1924 -1928 -1932 -1936 -1940 -1944 -1948 -1952 -1956 -1960 -1964 -1968 -1972 -1976 -1980 -1984 -1988 -1992 -1996 -2000 -2004 -2008 -2012 -2016 -2020 -2024 -2028 -2032 -2036 -2040 -2044 -2048 -2052 -2056 -2060 -2064 -2068 -2072 -2076 -2080 -2084 -2088 -2092 -2096 -2100 -2104 -2108 -2112 -2116 -2120 -2124 -2128 -2132 -2136 -2140 -2144 -2148 -2152 -2156 -2160 -2164 -2168 -2172 -2176 -2180 -2184 -2188 -2192 -2196 -2200 -2204 -2208 -2212 -2216 -2220 -2224 -2228 -2232 -2236 -2240 -2244 -2248 -2252 -2256 -2260 -2264 -2268 -2272 -2276 -2280 -2284 -2288 -2292 -2296 -2300 -2304 -2308 -2312 -2316 -2320 -2324 -2328 -2332 -2336 -2340 -2344 -2348 -2352 -2356 -2360 -2364 -2368 -2372 -2376 -2380 -2384 -2388 -2392 -2396 -2400 -2404 -2408 -2412 -2416 -2420 -2424 -2428 -2432 -2436 -2440 -2444 -2448 -2452 -2456 -2460 -2464 -2468 -2472 -2476 -2480 -2484 -2488 -2492 -2496 -2500 -2504 -2508 -2512 -2516 -2520 -2524 -2528 -2532 -2536 -2540 -2544 -2548 -2552 -2556 -2560 -2564 -2568 -2572 -2576 -2580 -2584 -2

Values of geopotential height (Z), and acceleration due to gravity (g) at 40° latitude for geometric height (z)

z (km)	Z (km)	g (ms⁻²)
0	0	9.81
1	1.00	9.80
10	9.99	9.77
100	98.47	9.5
500	463.6	8.43

High Pressure

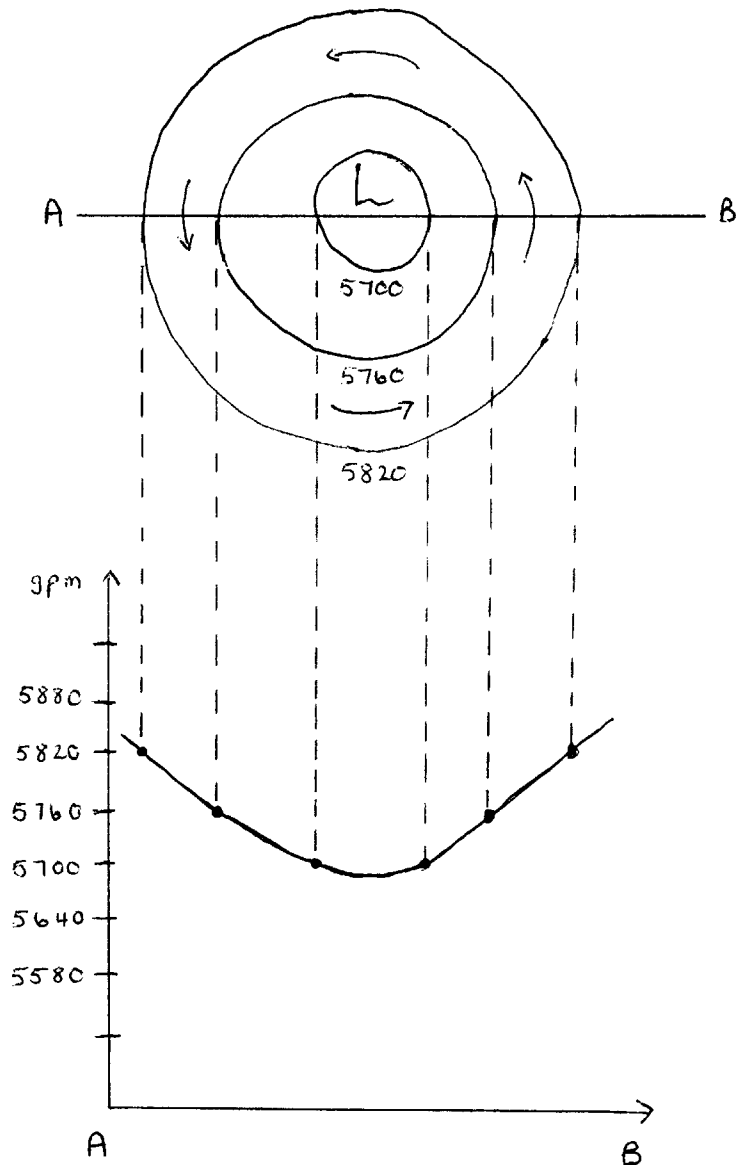


Plan view of the geopotential heights of the 500mb pressure surface

Vertical C/S along the line AB shown in the plan view – shows the height of the 500mb pressure surface.

Pressure along the solid line is 500 mb. Pressure below this line is greater than 500 mb and pressure above this line is greater than 500 mb

Low Pressure



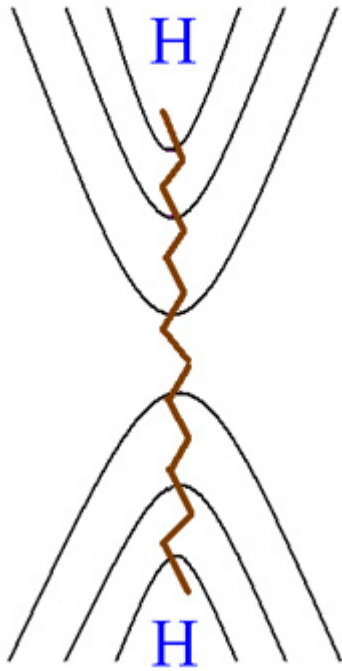
Plan view of the geopotential heights of the 500mb pressure surface

Vertical C/S along the line AB shown in the plan view – shows the height of the 500mb pressure surface.

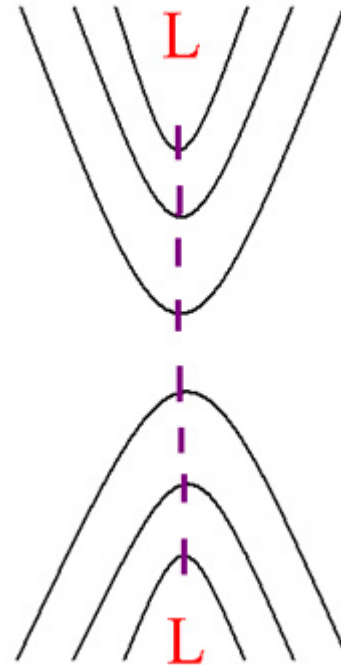
Pressure along the solid line is 500 mb. Pressure below this line is greater than 500 mb and pressure above this line is greater than 500 mb

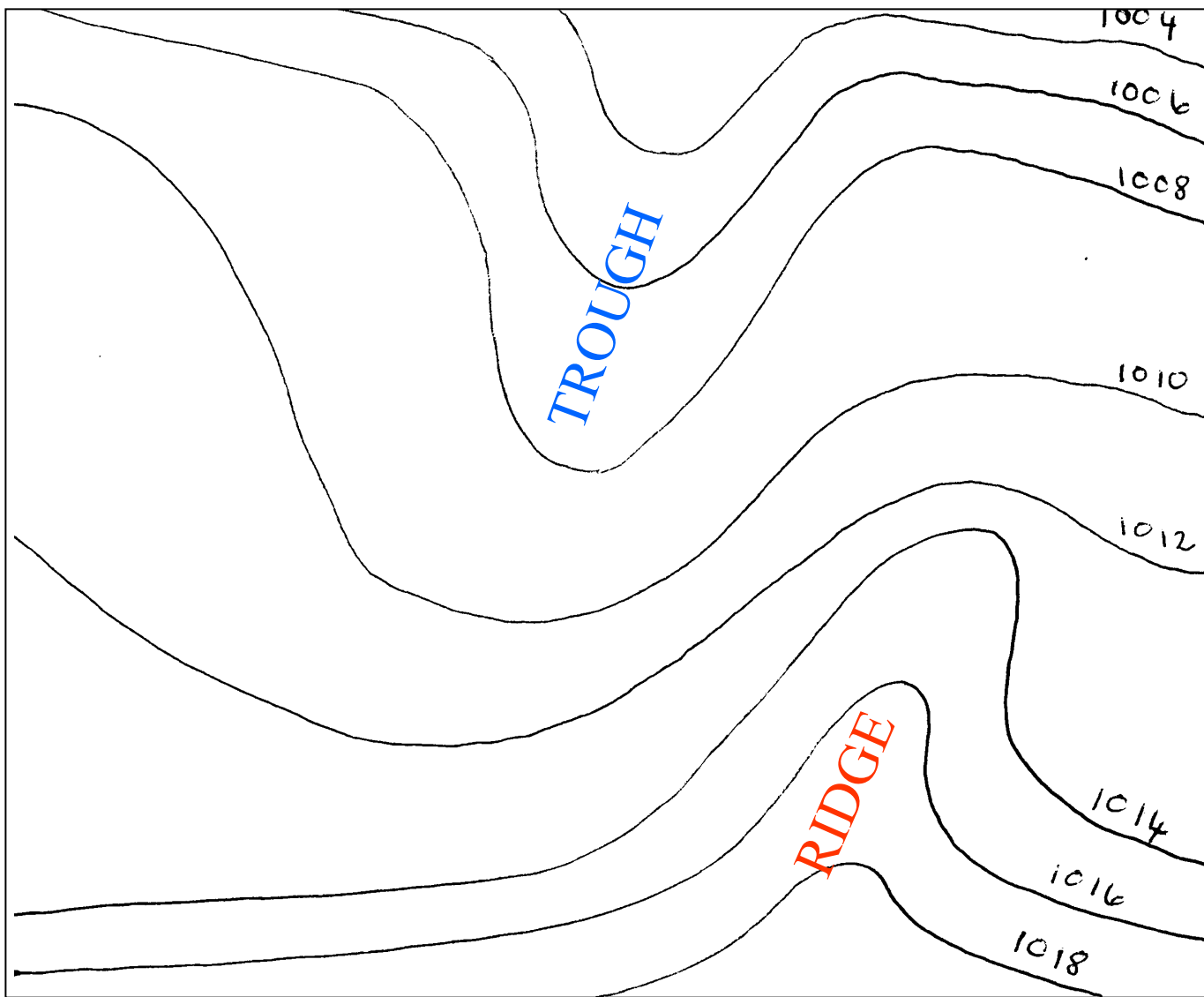
Troughs and Ridges

Ridge: an extended area of high pressure

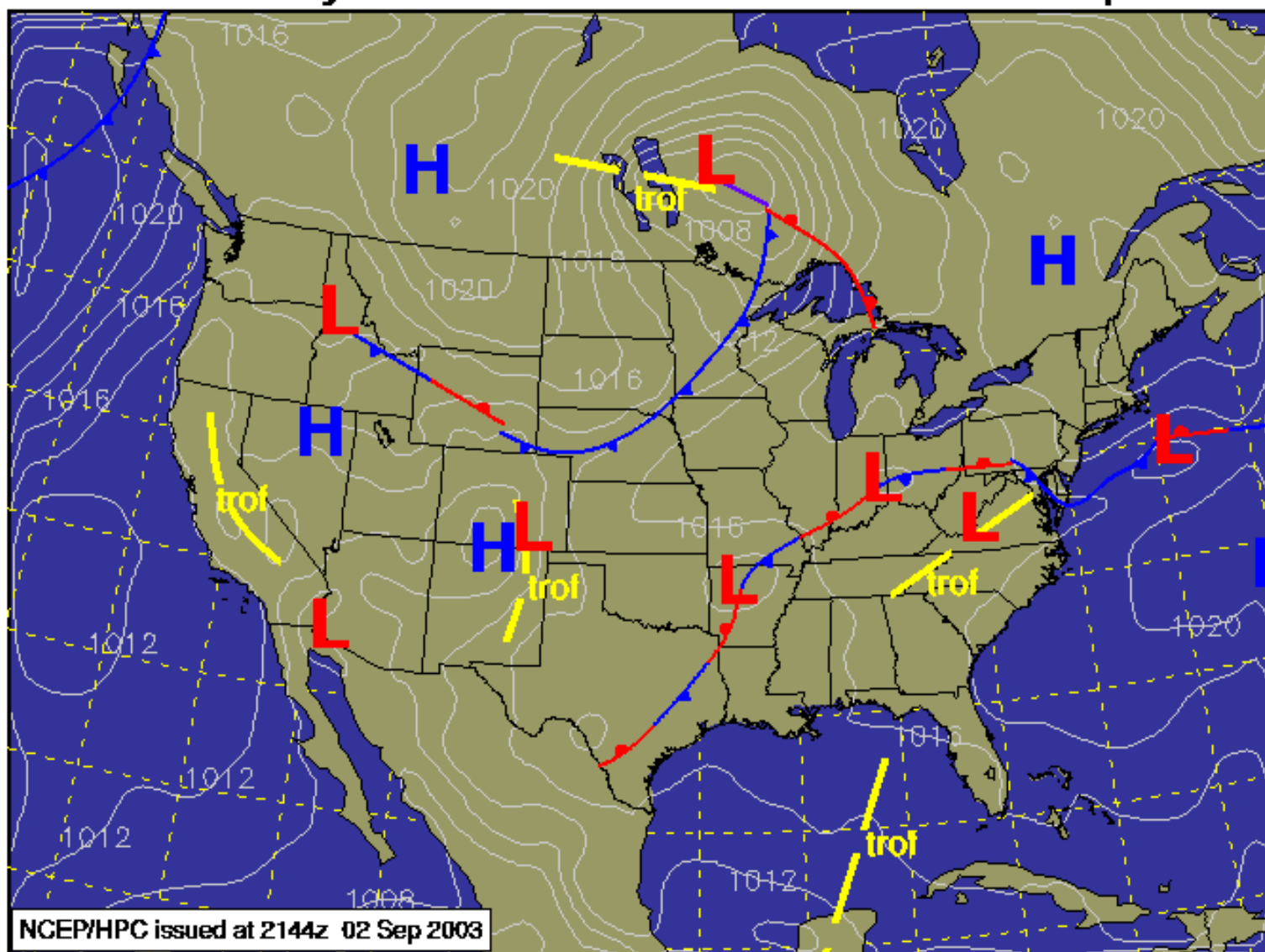


Trough: an extended area of low pressure





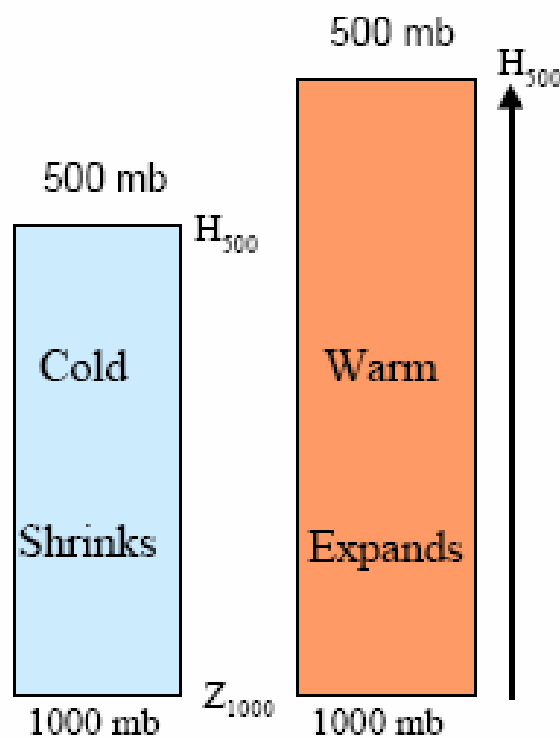
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3.4

Thickness

Hydrostatic equilibrium



- Hydrostatic equilibrium is when the downward gravity force (weight) exactly balances the upward PGF
- For a cold air mass, the rate of pressure decrease with height is largest
 - *Cold air mass means a shorter column (lower 500 mb height)*

- Useful to express Z in terms of temperature and pressure:

$$d\phi = g dz = -\alpha dp$$

From the equation of state :

$$p = \rho R_d T_v \quad \Rightarrow \quad \rho = p / R_d T_v$$

Substituting into the hydrostatic eqn :

$$\frac{dp}{dz} = -\rho g = -\frac{pg}{R_d T_v} \quad \Rightarrow \quad dp = -\frac{pg}{R_d T_v} dz$$

But $d\phi = g dz$

$$\therefore dp = -\frac{p}{R_d T_v} d\phi \quad \Rightarrow \quad -\frac{R_d T_v}{p} dp = d\phi$$

$$-\frac{R_d T_v}{p} dp = d\phi$$

Integrating from p_1 to p_2 with geopotentials ϕ_1 to ϕ_2

$$\phi_2 - \phi_1 = -R_d \int_{p_1}^{p_2} T_v \frac{dp}{p} = -R_d \int_{p_1}^{p_2} T_v d \ln p$$

Dividing both sides by g_0 :

$$Z_2 - Z_1 = -\frac{R_d}{g_0} \int_{p_1}^{p_2} T_v d \ln p$$

Replacing T_v with an average $\bar{T}_v$ of the layer :

$$Z_2 - Z_1 = -\frac{R_d \bar{T}_v}{g_0} \ln \left(\frac{p_2}{p_1} \right) = \frac{R_d \bar{T}_v}{g_0} \ln \left(\frac{p_1}{p_2} \right)$$

$$Z_2 - Z_1 = \frac{R_d \bar{T}_v}{g_0} \ln \left(\frac{p_1}{p_2} \right)$$

- This is the **Thickness Equation or Hypsometric Equation**
- The thickness of the layer between any two pressure surface is proportional to the mean virtual temperature of the layer
- As $\bar{T}_v$ increases, the air expands and the layer becomes thicker
- The geopotential height of the 500 mb surface is:

$$Z_{500} = \frac{R_d \bar{T}_v}{g_0} \ln \left(\frac{p_{sea-level}}{p_{500}} \right)$$

- Z_{500} will be low if $p_{sea-level}$ is low for a fixed $\bar{T}_v$
- For a given $p_{sea-level}$, Z_{500} is low if $\bar{T}_v$ between surface and 500 mb is low
- If 3D distribution of $\bar{T}_v$ is known, as well as geopotential height of one surface => can get the geopotential height of the other pressure surface

Another view

From the hydrostatic equation and the ideal gas law :

$$\frac{\partial p}{\partial z} = -\rho g = -\frac{p g}{R_d T}$$

$$\Rightarrow \frac{\partial \ln p}{\partial z} = -\frac{g}{R_d T}$$

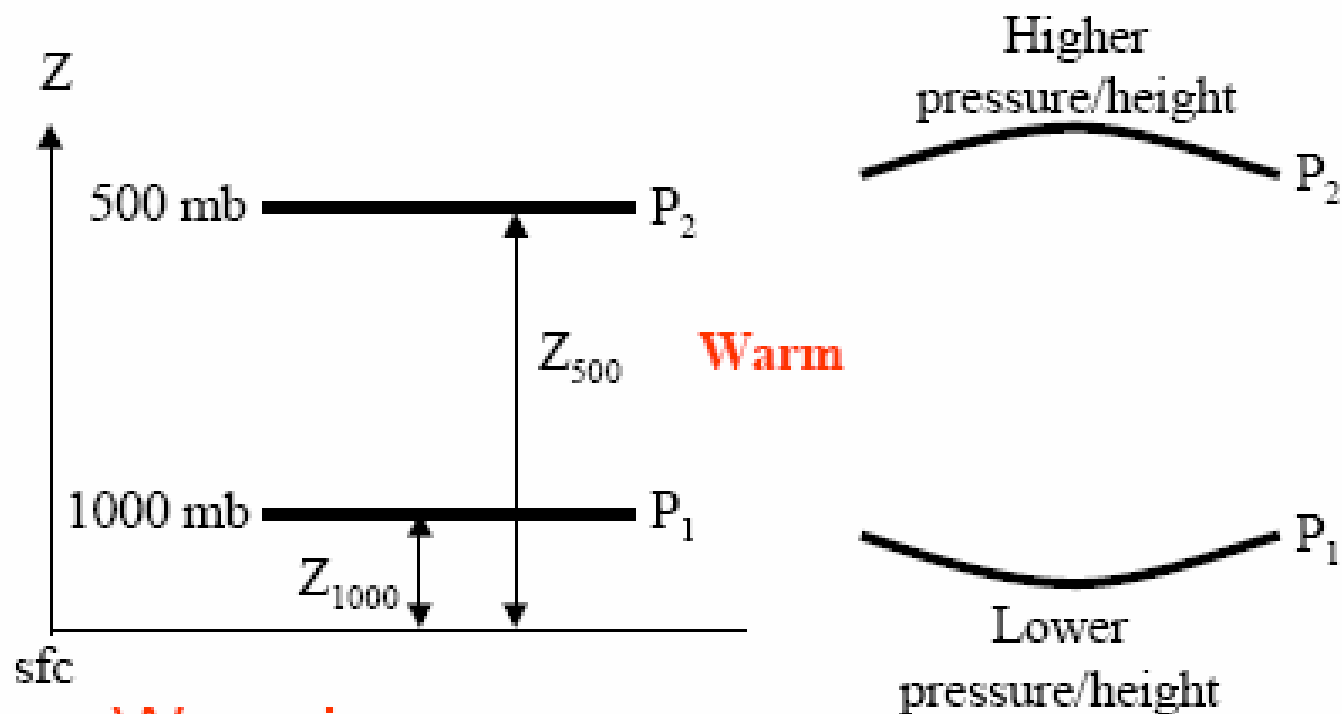
Integrating between two heights ($z_1, z_2; z_2 > z_1$) corresponding to two pressure surfaces ($p_1, p_2; p_1 > p_2$):

$$\int_{z_1}^{z_2} \frac{\partial \ln p}{\partial z} dz = \ln \frac{p_1}{p_2} = -\frac{g}{R_d} \int_{z_1}^{z_2} \frac{dz}{T} = -\frac{g}{R_d} (z_2 - z_1) \left(\overline{\frac{1}{T}} \right)$$

where the mean value theorem of calculus has been used to remove $1/T$ from the integrand.

Assuming that $\left(\overline{1/T} \right) \cong 1/\bar{T}$ and rearranging gives :

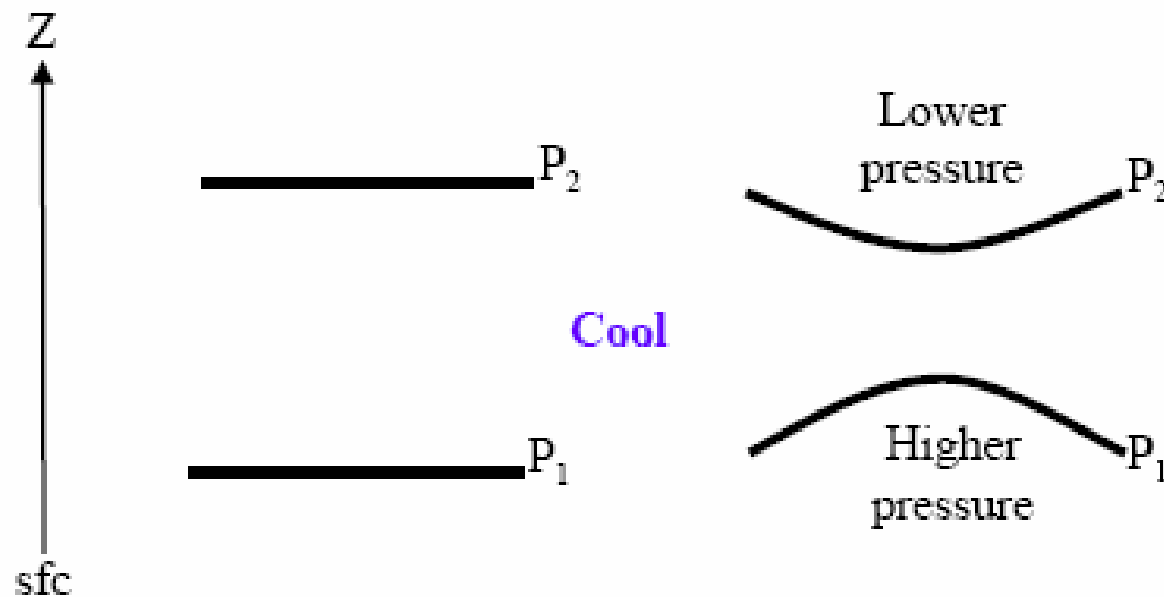
$$(z_2 - z_1) = \Delta z = \frac{R_d \bar{T}_v}{g} \ln(p_1 / p_2)$$



■ Warming

- Press/height rises aloft, falls near surface
- anticyclonic aloft, cyclonic beneath

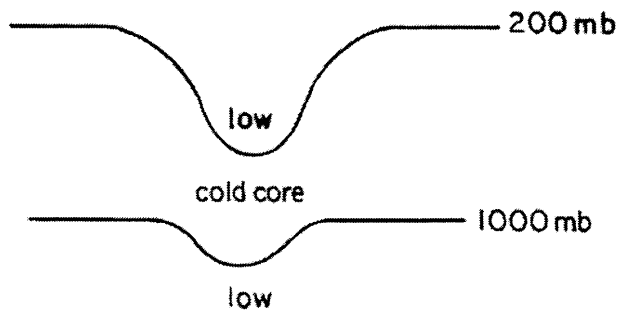
■ Pressure/height rising aloft and falling near surface $\Rightarrow$ warming



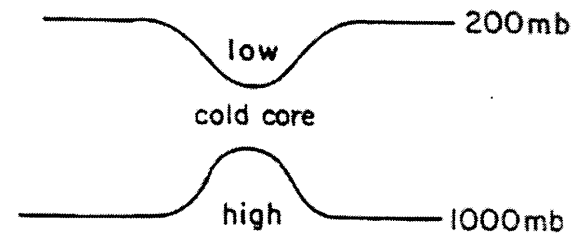
■ Cooling

- height/press falls aloft, rises near surface
- cyclonic flow aloft, anticyclonic beneath

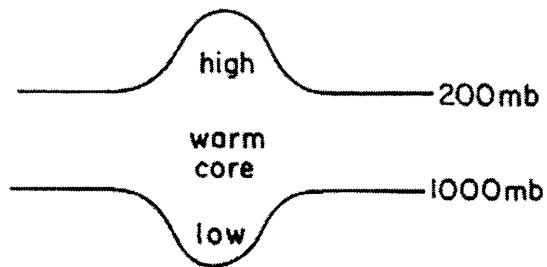
■ Pressure/height falling aloft and rising near surface $\Rightarrow$ cooling



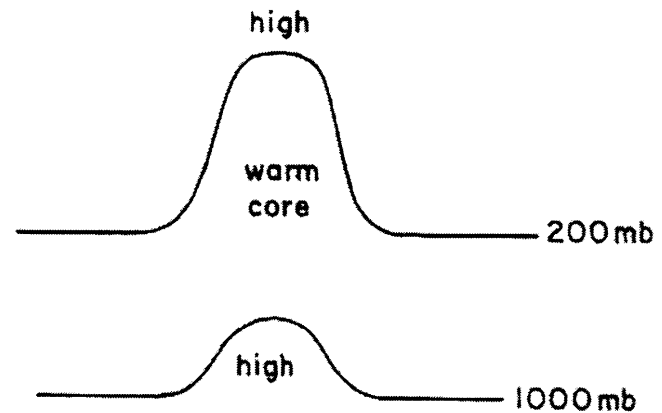
Winter Storm
Increases Intensity
with Height



Arctic High
Decreases Intensity
with Height



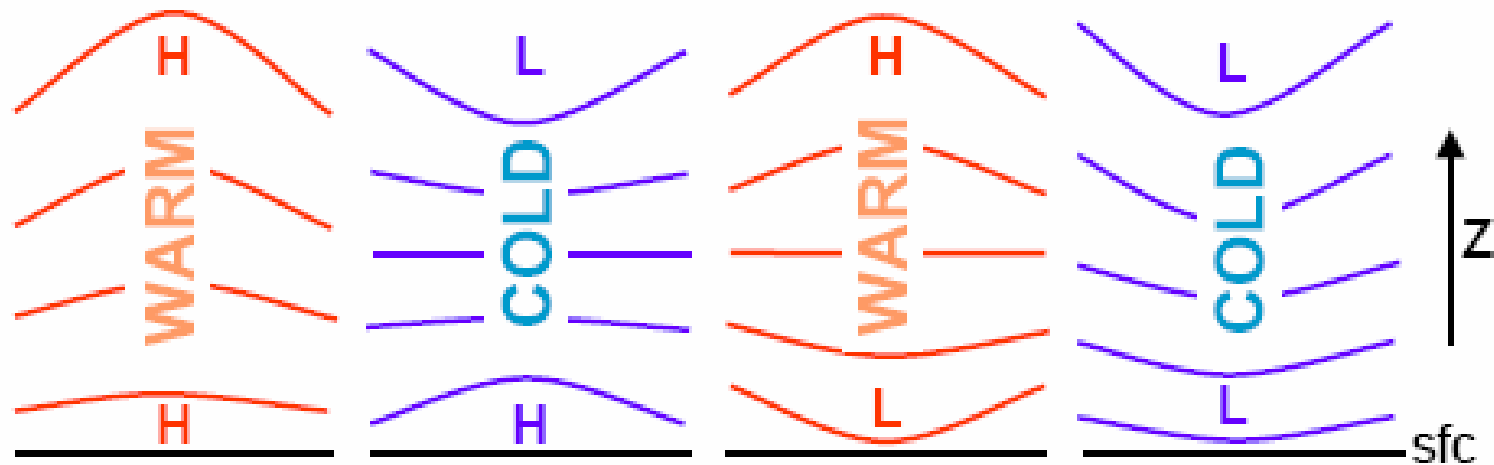
Hurricane
Decreases Intensity
with Height



Subtropical High
Increases Intensity
with Height

- The subtropical anticyclone (warm core system) and the extratropical cyclone (cold core system) increase in intensity with height => systems better defined in upper troposphere than at surface
- The tropical cyclone (warm core system) and the polar anticyclone (cold core system) become less intense with height

Cross sections of pressure for cold and warm systems



Warm core high

Strong high aloft

Weaker surface
high

Example: Summer
heat wave

Cold core high

Trough aloft

High surface

Example:
Siberian high,
high that forms
behind cold front
in winter

Warm core low

Ridge aloft

Low surface

Example:
Thermal heat
low over
desert

Cold core low

Strong trough aloft

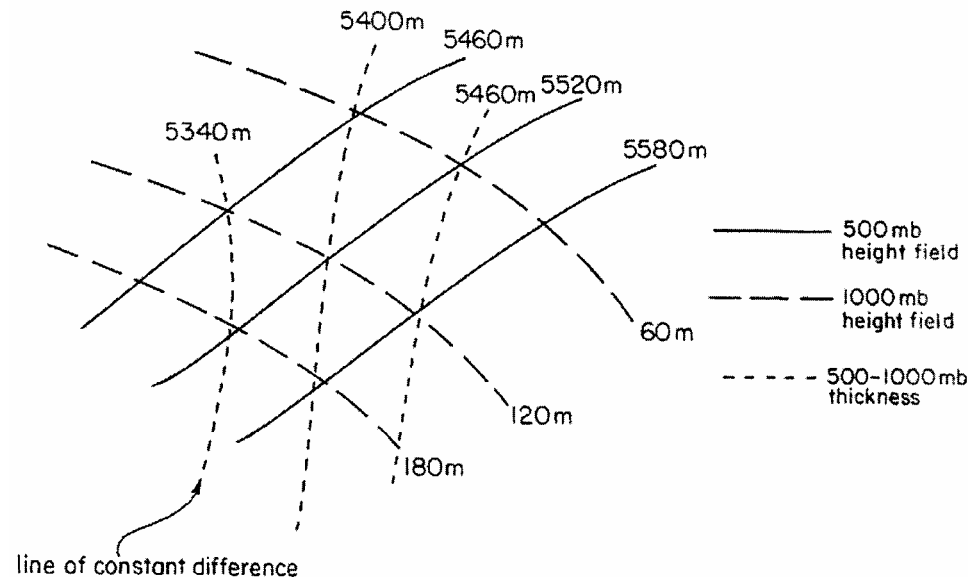
Weaker surface low

Example: Occluded
midlatitude storm

Implications

- Cold air $\Rightarrow$ low press aloft, high at sfc
 - 300/500 mb trough above cold regions
 - 300 mb cyclonic vortex (low) above poles, most intense in winter when coldest
 - Polar high at surface
- Warm air - high pressure aloft, low sfc
 - Equator - 300 mb high, surface trough
 - Upper level winds anticyclonic above warm regions
 - Anticyclone above intense convection such as tropical cyclone or monsoon trough

- Thickness can be determined graphically given two constant pressure analyses:



Two rules: include all intersections of the two constant pressure height fields and do not cross contours

Some Uses of Thickness

- Guideline for precipitation type: rain vs snow
- Warm or cold air advection
- Thickness analyses are used to locate synoptic fronts and to determine their intensity (by definition a synoptic front must be associated with a horizontal thickness gradient)

Thickness and Precipitation Type

- Thickness used to characterize the weather – often used to determine the 50% probability of snow ($\Delta z|_{\text{rain/snow}}$) given that precipitation is occurring – also referred to as the rain-snow line

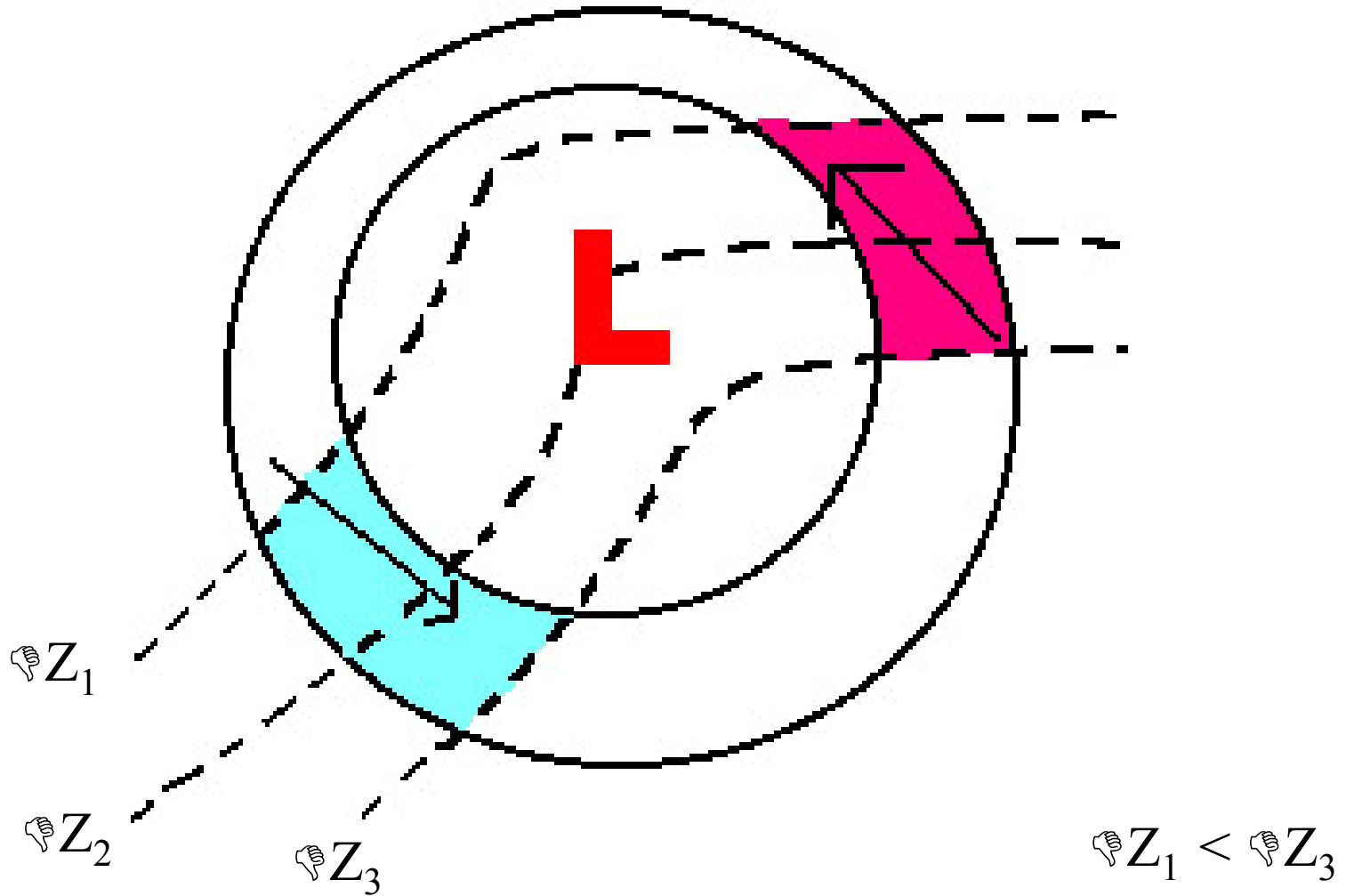
- The table shows thickness values associated with a 50% chance of snow given that precipitation is occurring

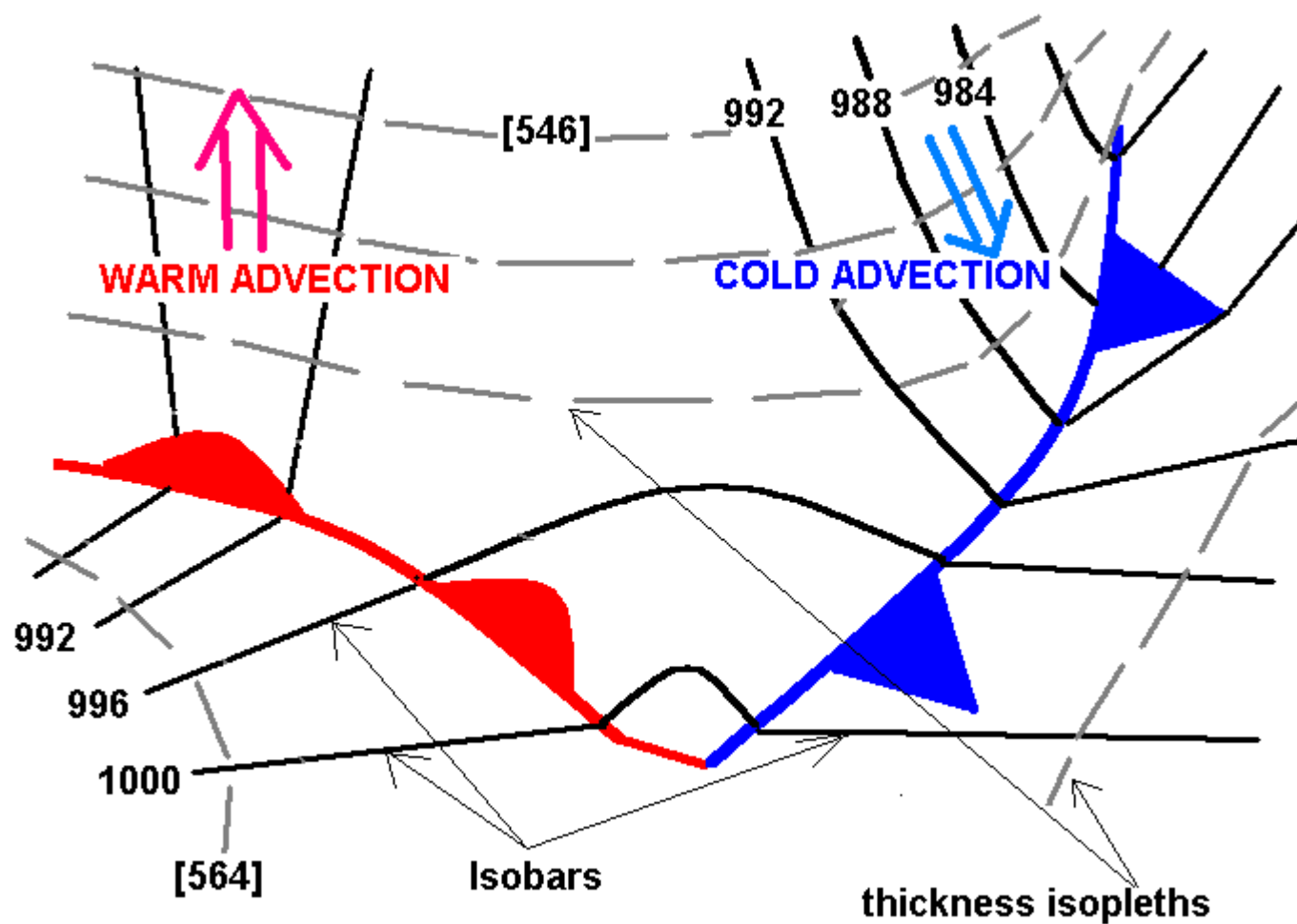
	$\Delta z _{\text{rain/snow}}$	
Bismark, North Dakota	5413	
Bristal, Tennessee	5387	
Baltimore, Maryland	5362	← low
Beckley, West Virginia	5426	
Chicago, Illinois	5391	
Columbia, Missouri	5400	
El Paso, Texas	5467	
Goodland, Kansas	5461	
Denver, Colorado	5501	← high
Cheyenne, Wyoming	5509	
Grand Junction, Colorado	5454	
Havre, Montana	5410	
Lander, Wyoming	5501	
Medford, Oregon	5262	
Minneapolis, Minnesota	5408	
Missoula, Montana	5396	
Norfolk, Virginia	5371	
Philadelphia, Pennsylvania	5361	
Pueblo, Colorado	5489	
Reno, Nevada	5427	
Seattle-Tacoma, Washington	5205	

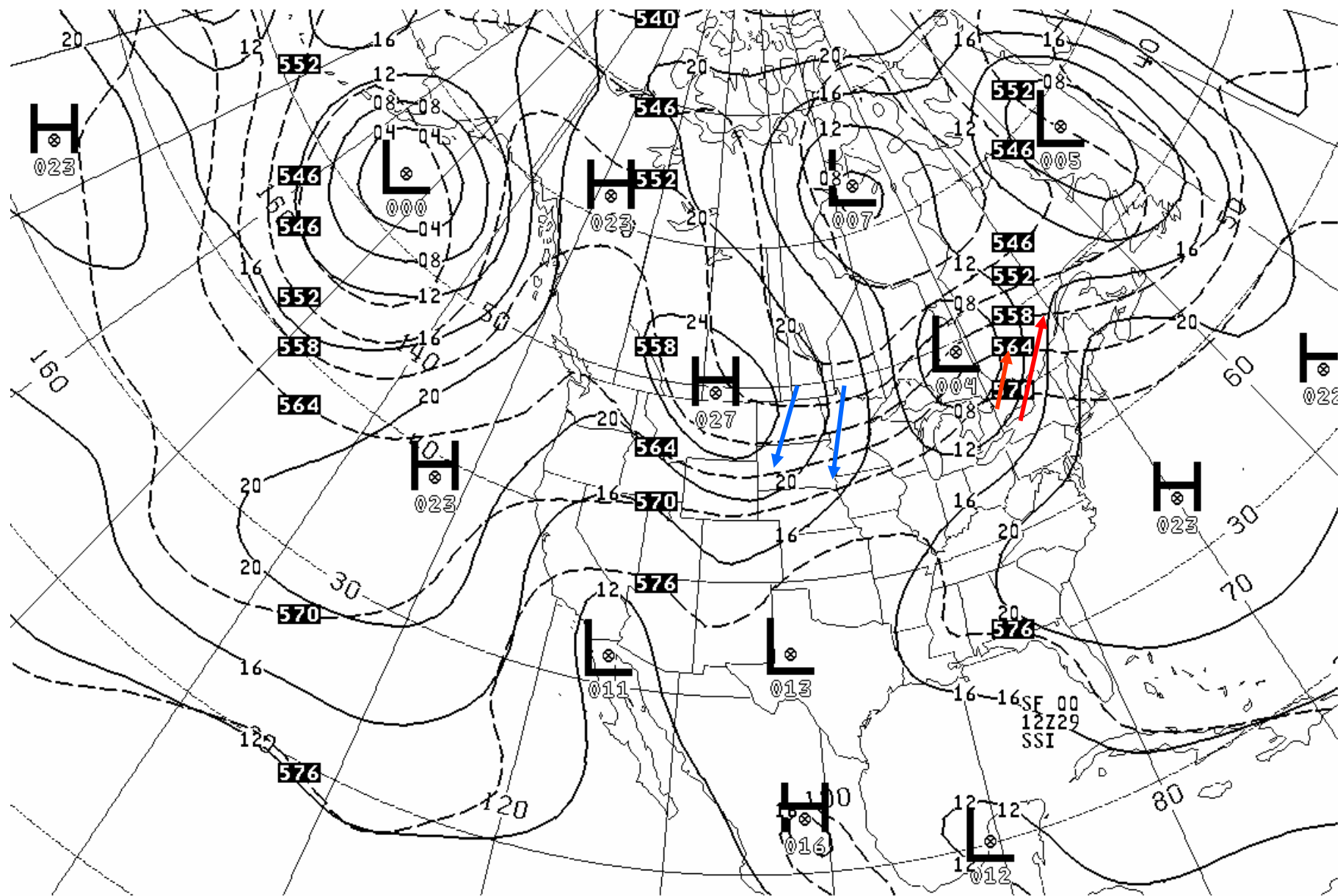
Examples

- In eastern US near sea level, $\Delta z = 5400$ m for the 1000 to 500 mb layer (mean $T = -1.6^{\circ}\text{C}$) closely corresponds to the 50% probability between rain and snow if precipitation is occurring
- Generally the 5400m line is used as an indicator of the rain-snow line
- $\Delta z|_{\text{rain/snow}}$ is substantially higher for sites at relatively high altitudes like Denver and Cheyenne - only a portion of the layer exists – generally the warmest part of the layer does not exist

Thickness and Cold and Warm Air Advection



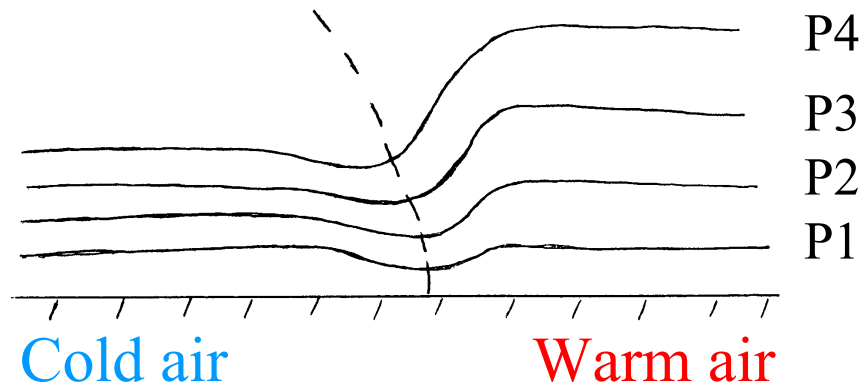
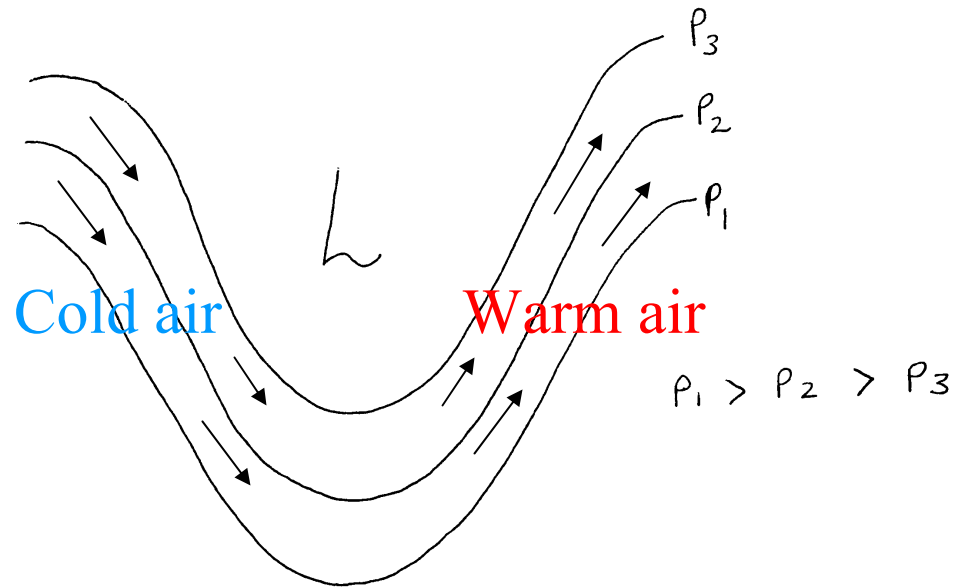




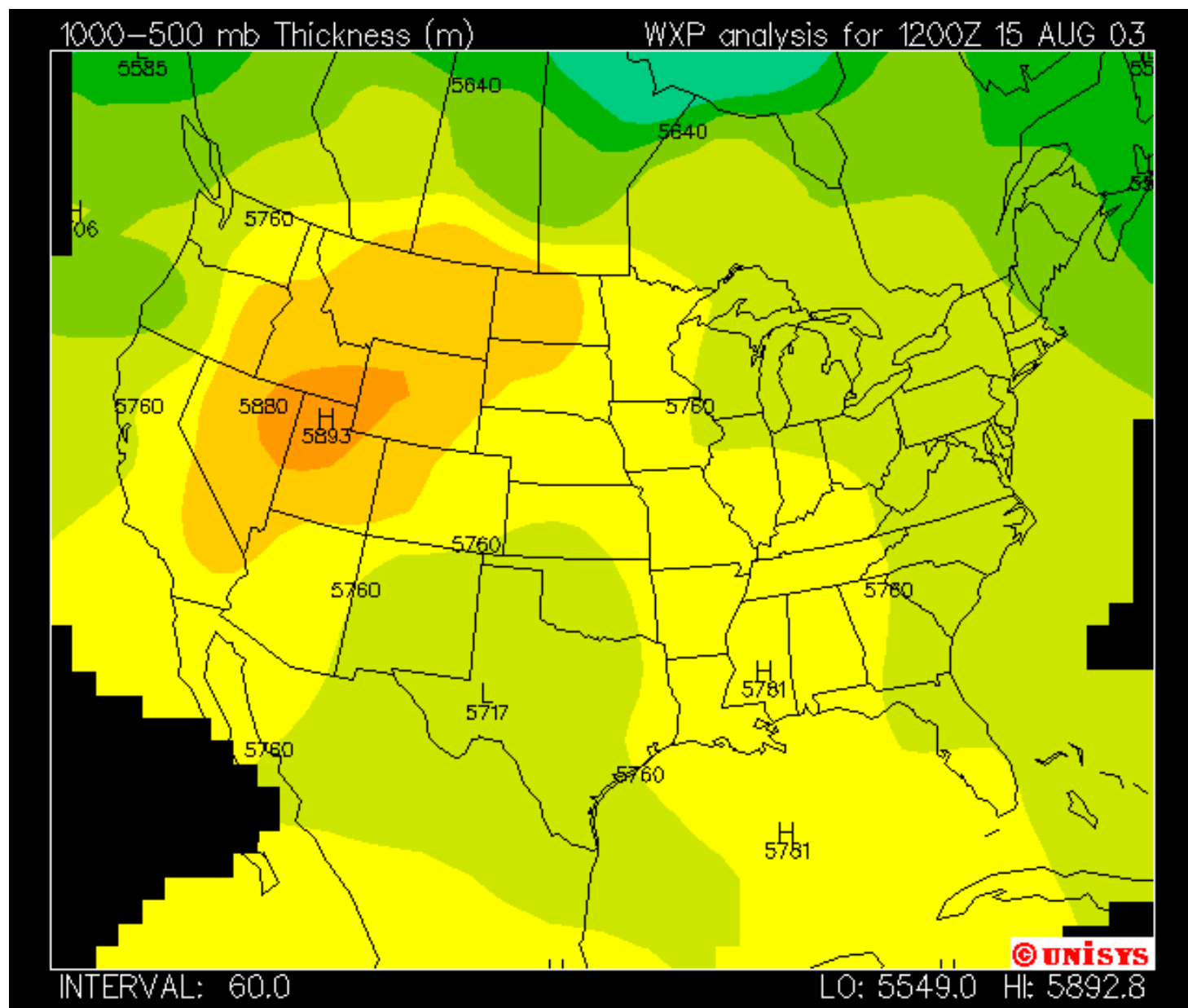
D190 ^ ANALYSIS SFC/1000-500MB THICKNESS VALID 12Z FRI 29 AUG 2003

Baroclinic Wave:

- Coldest air and lowest thickness to the west of the surface low
- System slopes westward with height



Thickness and Fronts



Surface data plot for 16Z 15 AUG 03

This meteorological plot displays surface data for August 15, 2003, at 16Z. The map covers the Gulf of Mexico and surrounding land areas. A low-pressure system (L) is centered over the Gulf, with a cold front (blue line with triangles) extending north and a warm front (red line with semicircles) extending northeast. High-pressure systems (H) are located to the west and south of the low. The plot includes numerous numerical data points representing temperature, pressure, and other meteorological variables across the region. The UNISYS logo is visible in the bottom right corner.

Intensities (Dbz): 20 30 40 45 50 55 Fronts at 15Z

Fronts at 15Z

A Basic Introduction to Fronts

- A front is a boundary between two different air masses
- Fronts are found in troughs of low pressure
- Winds tend to converge in the vicinity of fronts which leads to clouds and precipitation
- With fronts: **COLD AIR RULES**
 - The movement of the cold air determines the type of front

Fronts (cont)

- A front on the synoptic scale is characterized by the following 3 criteria:
 1. A front separates different air mass types
 2. A thickness gradient gives the approximate location of the front, which is located near the warm side of the gradient
 3. The type of front depends on the movement of the COLD air

Fronts (cont)

- At the surface, the following are used locate the position of the front more precisely (once the conditions above are met):
 1. A wind shift line
 2. A pressure trough
 3. A temperature discontinuity
 4. A dewpoint temperature discontinuity
 5. The pressure tendency pattern
 6. Horizontal variations in visibility
 7. Horizontal variations in precipitation type

Fronts (cont)

- There are 5 basic fronts:

1. Cold front

- at a cold front cold air is displacing the warmer air ahead of the front
- winds tend to blow into the front from the cold side of the front
- shown on surface maps as blue lines with blue triangles which point in the direction of the warmer air

2. Warm front

- At a warm front, cold air is retreating on the cold side of the front and is being replaced by warmer air
- Winds tend to blow away from the front
- Shown on surface maps as red lines with red semicircles which point in the direction of the colder air

Fronts (cont)

3. Stationary front

- Stationary fronts move very little or not at all as winds on the colder side of the front tend to blow parallel to the front
- Shown on surface maps as alternating blue triangles and red semicircles (blue triangles point in the direction of the warmer air)

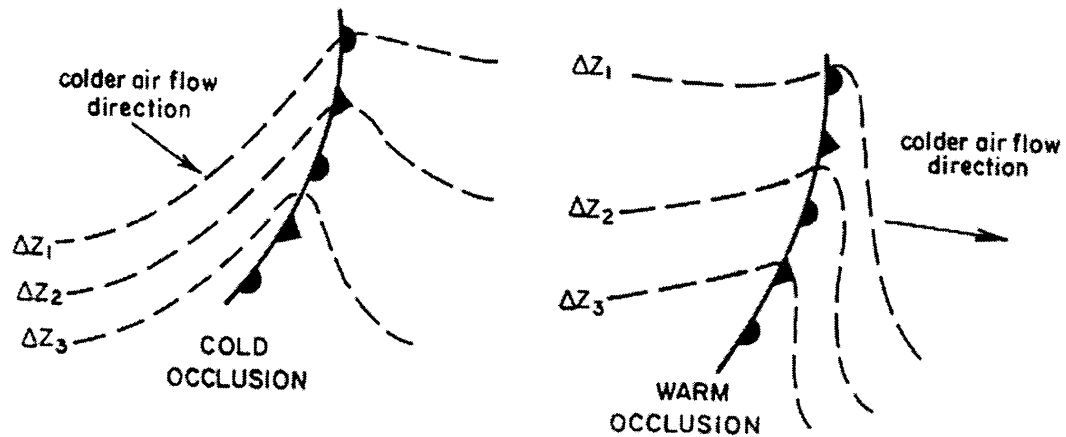
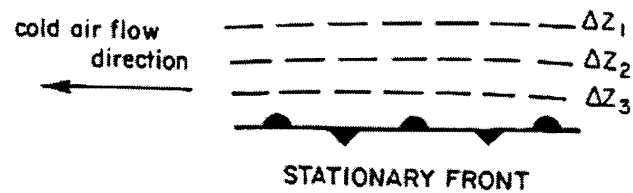
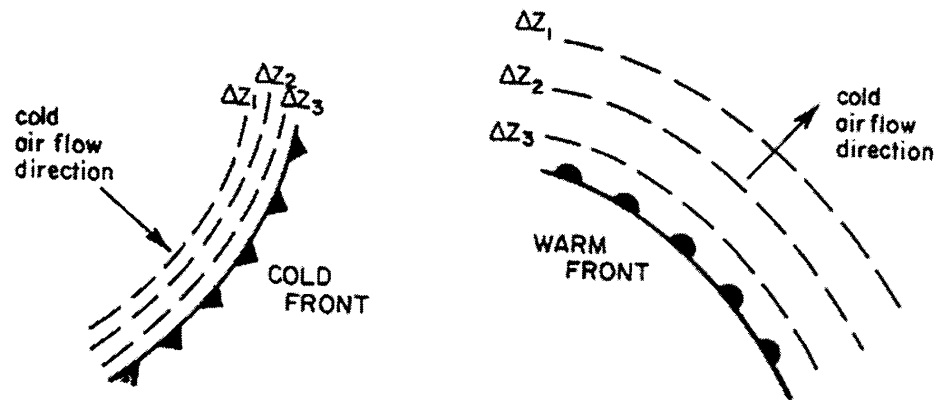
4. Cold Occlusion

- This type of front occurs when the cold front catches up with and overtakes the warm front
- The air behind the cold front is colder than the air ahead of the warm front
- Shown on surface maps as purple with alternating triangles and semicircles on the same side of the line pointing in the direction of movement of the front

Fronts (cont)

5. Warm Occlusion

- This type of front occurs when the cold front catches up with and overtakes the warm front
- The air behind the cold front is warmer than the air ahead of the warm front
- Shown on surface maps as purple with alternating triangles and semicircles on the same side of the line pointing in the direction of movement of the front

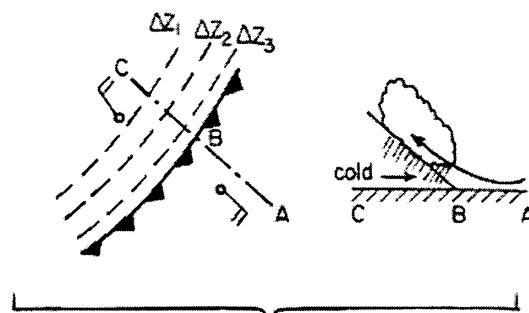


$$\text{👉} Z_1 < \text{👉} Z_3$$

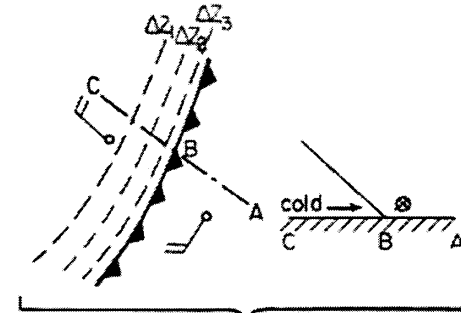
Fronts (cont)

There are two types of each front:

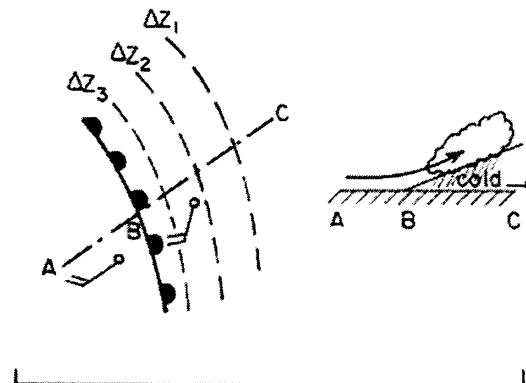
1. Active fronts: occurs when the warmer air mass is overrunning the cold air mass
2. Inactive fronts: occurs when the warmer air mass is not overrunning the cold air mass



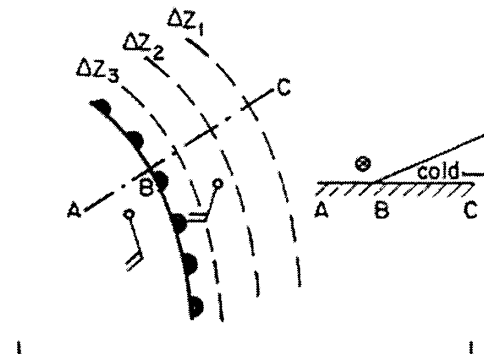
Active Cold Front



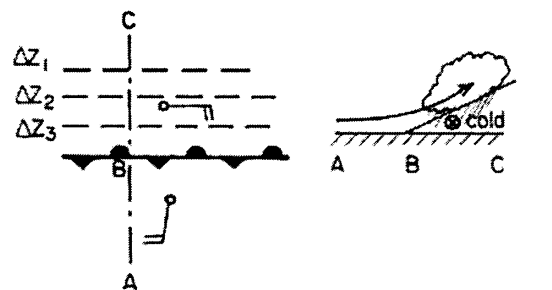
Inactive Cold Front



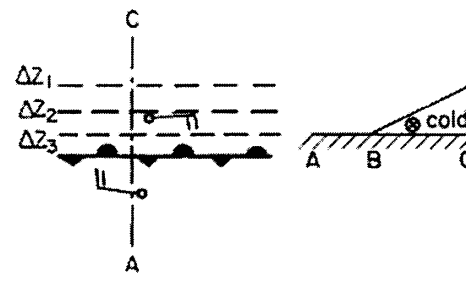
Active Warm Front



Inactive Warm Front

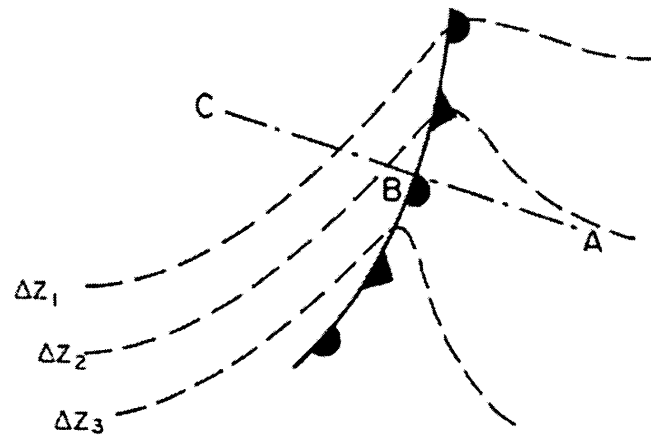


Active Stationary Front

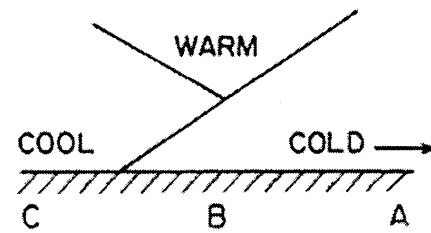
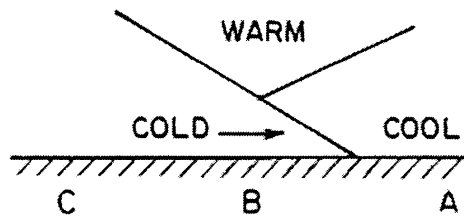
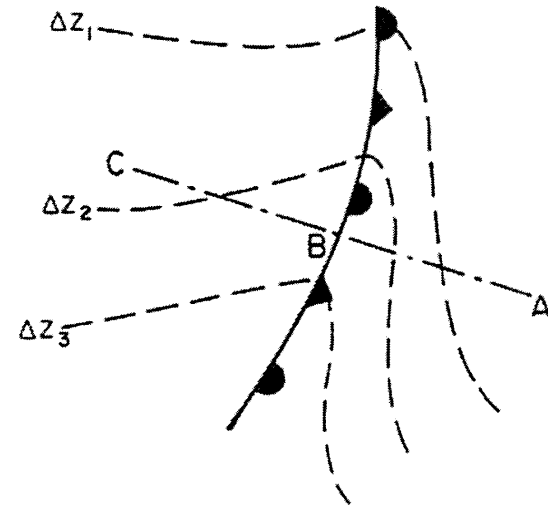


Inactive Stationary Front

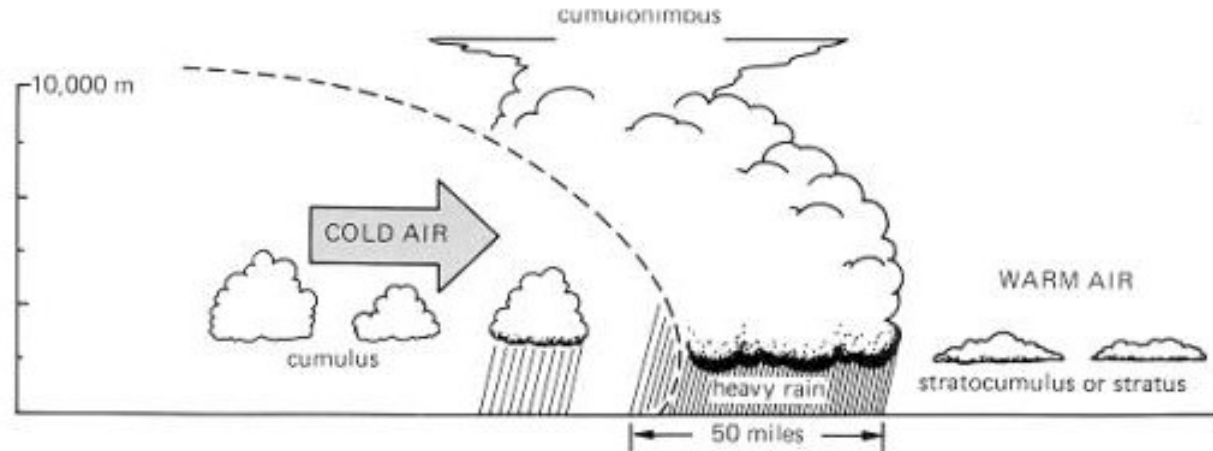
Cold Occlusion



Warm Occlusion

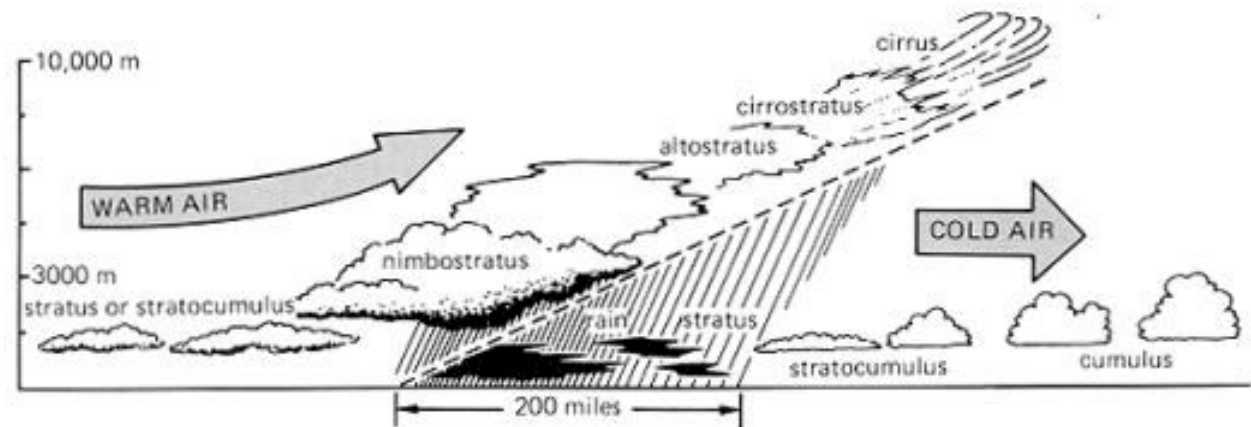


THE LIFE OF A DEPRESSION



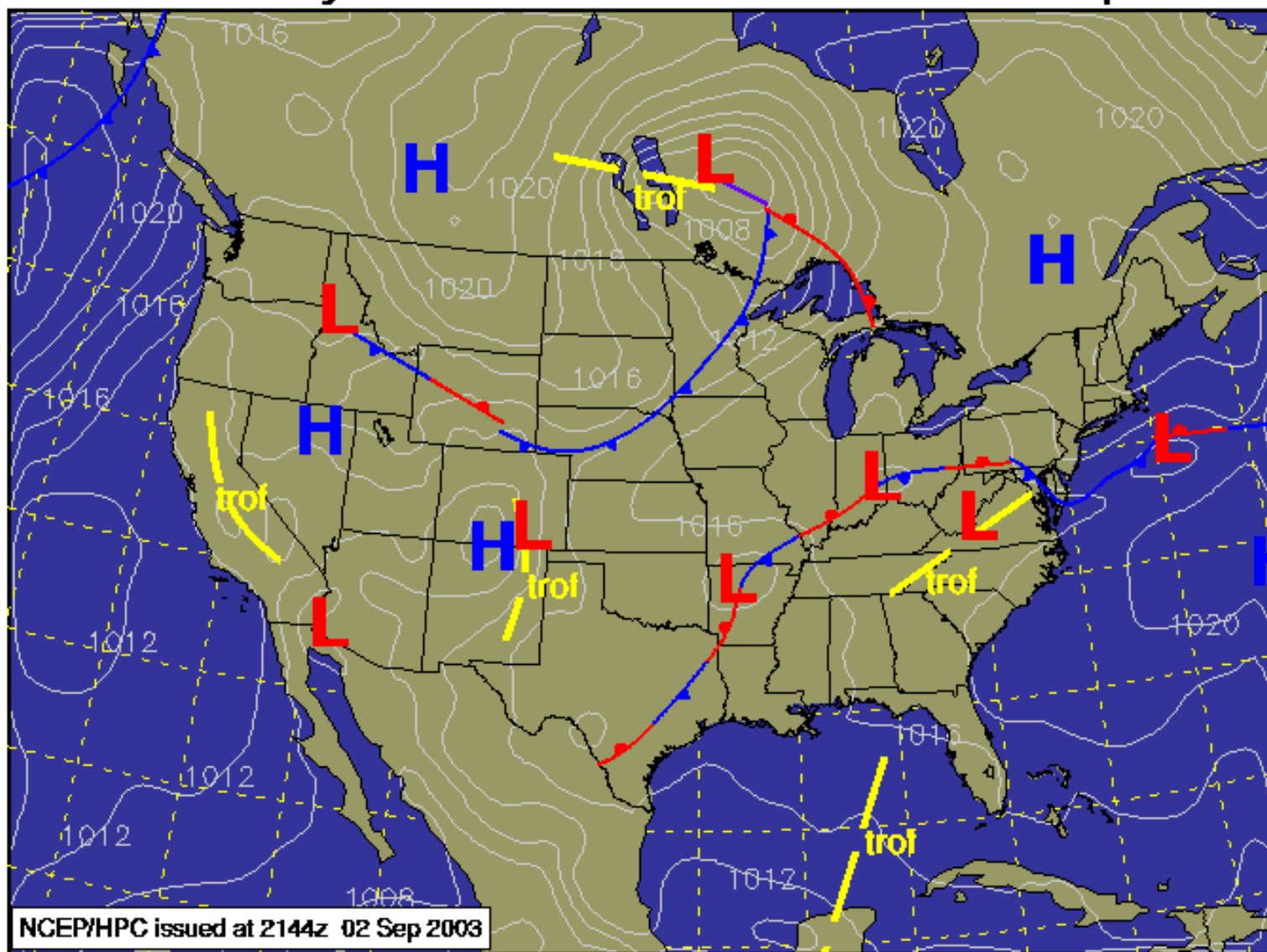
C/S through
an active cold
front

THE LIFE OF A DEPRESSION

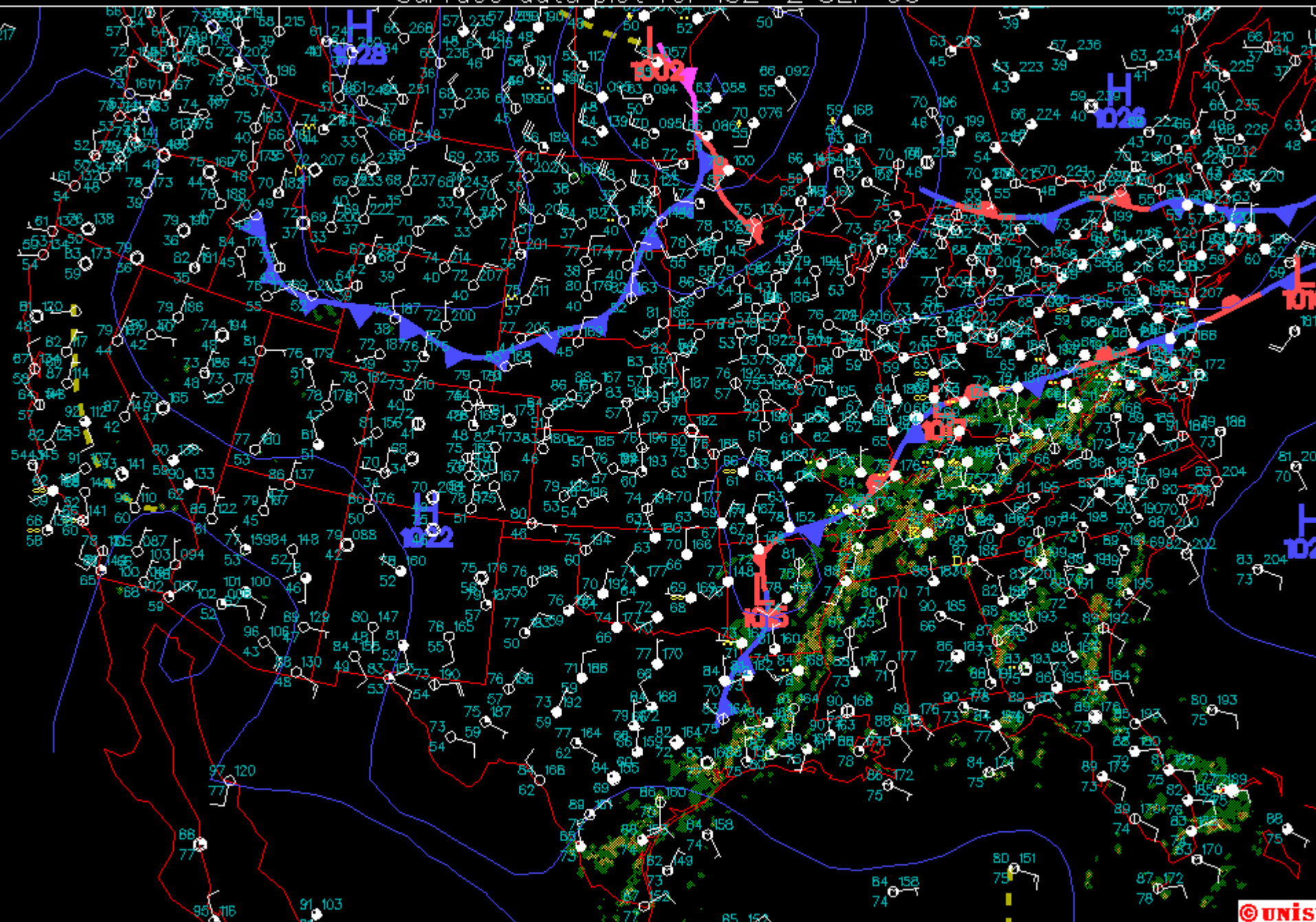


C/S through an
active warm
front

Analysis valid 2100 UTC Tue 02 Sep 2003



Surface data plot for 18Z 2 SEP 03



pressures (Dbz): 20 30 40 45 50 55

Fronts at 15Z

3.5 First Law of Thermodynamics

- Systems may have macroscopic kinetic and potential energy
- Systems may also have **internal energy** due to kinetic and potential energy of its molecules
- Increases in internal energy in the form of molecular motions are manifested as increases in temperature

Consider a unit mass of gas which takes in a certain quantity of heat q (joules) by radiation or thermal conduction

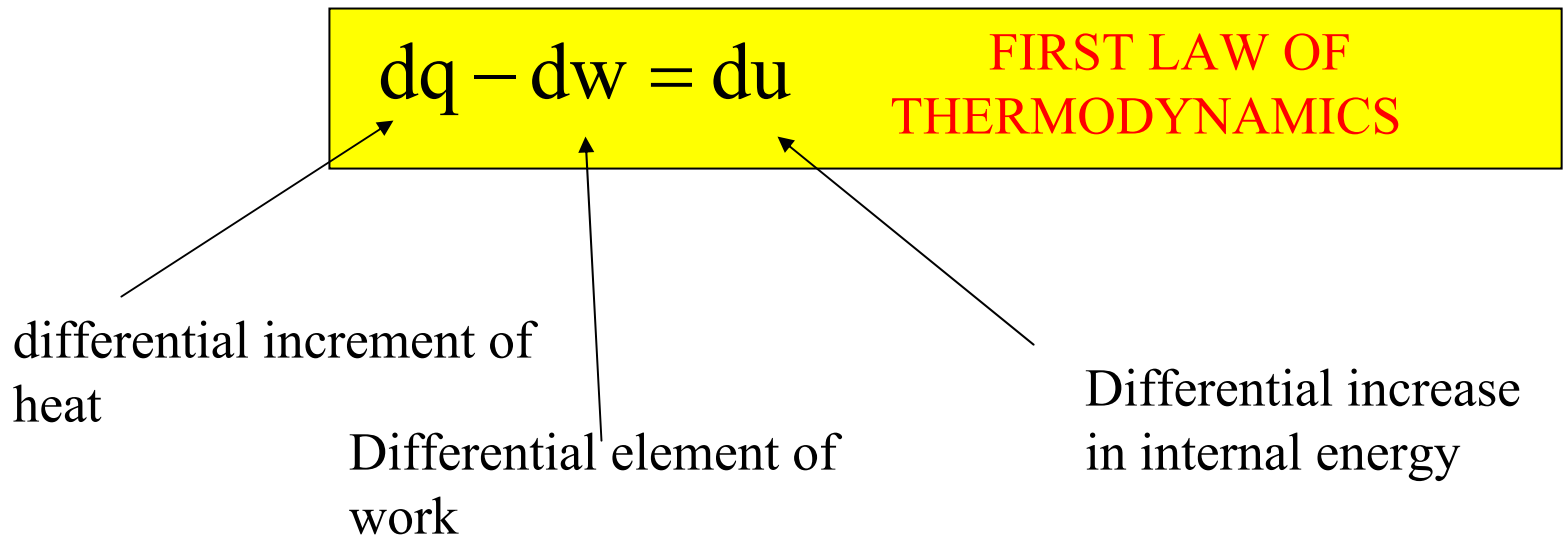
As a result, the gas may do a certain amount of external work w (joules)

The excess of the energy supplied to the gas over and above the external work done by the gas is:

$$q - w = u_2 - u_1$$

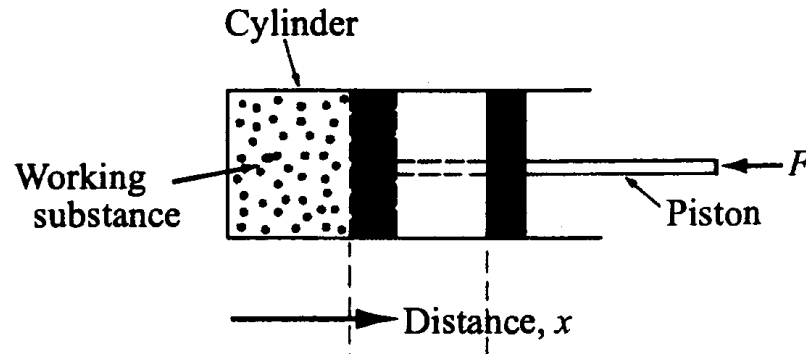
Where u_1 and u_2 are the internal energies of the gas before and after the change

In differential form:



State Variables

- The change in internal energy du depends only on the initial and final states of the gas
- du is therefore independent of the manner by which the gas is transferred between these two states
- Such parameters, like u , are called **functions of state**
- Neither heat q nor work w are functions of state as their values depend on HOW a gas is transformed from one state to another



Consider a gas contained in a cylinder of fixed cross-sectional area with a moveable, frictionless piston

The volume V of the gas is proportional to the distance d from the base of the cylinder as the cross-sectional area is constant: $V \propto d$

If the piston is moved outwards through an incremental distance dx , while its pressure remains essentially constant, the work dw done by the gas in expanding is equal to the force exerted by the piston multiplied by the distance dx :

$$dw = Fdx$$

but pressure = force / unit area

$$\Rightarrow p = F/A$$

$$\Rightarrow F = pA$$

$$\therefore dw = pAdx$$

but $Adx = dV$

$$\therefore dw = pdV$$

- The work done by the gas when its volume increases by a small amount is equal to the pressure of the gas multiplied by the increase in volume of the gas
- pdV is called p-V work
- For a unit mass of gas we have: $dw = pd\alpha$

$$\Rightarrow dq = du + dw = du + pd\alpha$$

First Law of Thermodynamics

Joule's Law

- Joule showed following a series of lab experiments that when a gas expands without doing external work, by expanding into a vacuum, and without taking in or giving out heat, that the temperature of the gas does not change
- Implications:
 - If a gas does not do external work: $dw=0$
 - If a gas does not take in or give out heat: $dq=0$
 - From the first law of thermodynamics $\Rightarrow du=0$
 - $\Rightarrow$ internal energy of a gas is a function of temperature only
 - $\Rightarrow$ the internal energy is independent of its volume if the temperature is kept constant

3.6 Specific Heats

- Definition: the amount of heat required to change the temperature of a unit mass of a substance by 1 degree
- Suppose a small amount of heat dq is given to a unit mass of gas causing its temperature to increase from T to $T+dT$ (without any phase changes occurring) then the ratio:

$$dq/dT$$

is called the specific heat of the gas

- Specific heat can take vary depending on how the gas changes as it receives the heat

• Specific Heat at Constant Volume

If the volume of the gas is kept constant, then we define the specific heat of the gas as :

$$c_v = \left(\frac{dq}{dT} \right)_{v \text{ const}}$$

But if the volume is constant :

$$dq = du + pd\alpha \xrightarrow{0} du$$

$$\therefore c_v = \left(\frac{du}{dT} \right)_{v \text{ const}}$$

Applying Joule's law for an ideal gas :

$$u = u(T) \text{ only}$$

$$\therefore c_v = \frac{du}{dT}$$

whether or not there is a change in volume

=>

$$dq = c_v dT + pd\alpha$$

First law of thermodynamics

- Specific Heat at Constant Pressure

If the pressure of the gas is kept constant we can define the specific heat as :

$$c_p = \left(\frac{dq}{dT} \right)_{p \text{ const}}$$

We showed above that

$$\begin{aligned} dq &= c_v dT + p d\alpha \\ &= c_v dT + d(p\alpha) - \alpha dp \end{aligned}$$

From the equation of state :

$$p\alpha = RT \Rightarrow d(p\alpha) = R dT$$

$$\therefore dq = c_v dT + R dT - \alpha dp = (c_v + R) dT - \alpha dp$$

If p is constant

$$dq = (c_v + R) dT$$

$$\Rightarrow \left(\frac{dq}{dT} \right)_{p \text{ const}} = c_v + R = c_p \quad \therefore \boxed{c_p = c_v + R}$$

and by substitution :

$$\boxed{dq = c_p dT - \alpha dp}$$

First law of thermodynamics

- $c_v = 717 \text{ J K}^{-1} \text{ kg}^{-1}$ for dry air
- $c_p = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$ for dry air
- $c_p = c_v + R \Rightarrow R = c_p - c_v = 287 \text{ J K}^{-1} \text{ kg}^{-1}$ which is the gas constant for dry air
- Why is c_p greater than c_v ?
 - when the volume is constant, no p-V work can be done
 - when the pressure is constant, a certain amount of the heat added to the gas will be used to do work as the gas expands against its environment
 - therefore more heat must be added to the gas to raise its temperature by a given amount if the pressure is constant compared to when the volume of the gas is kept constant

3.7

Enthalpy

- Define enthalpy as: $h = u + p\alpha$
 - enthalpy is defined at constant pressure
- As u , p and α are functions of state, h is a function of state

$$h = u + p\alpha$$

Differentiating : $dh = du + d(p\alpha)$

$$\Rightarrow dh = c_v dT + d(p\alpha)$$

We saw earlier that $dq = c_v dT + d(p\alpha) - \alpha dp$

$$\Rightarrow c_v dT + d(p\alpha) = dq + \alpha dp$$

Substituting : $dh = dq + \alpha dp$

$$\Rightarrow dq = dh - \alpha dp$$

Another form of the first law of thermodynamics

$$dq = dh - \alpha dp$$

But we showed earlier that : $dq = c_p dT - \alpha dp$

$$\therefore dh = c_p dT$$

or after integrating where h is taken as 0 when T = 0 :

$$h = c_p T$$

Previously we defined the geopotential as $d\phi = g dz = -\alpha dp$

Now $dq = dh - \alpha dp$

$$\Rightarrow dq = dh + d\phi$$

$$\Rightarrow dq = d(h + \phi) = d(c_p T + \phi)$$

Called the dry static energy or the Montgomery stream function

- If heat is neither added to or taken from a parcel in a hydrostatic atmosphere ($dq=0$): $dq = 0 \Rightarrow h + \phi = \text{const}$

$\Rightarrow$ the dry static energy is therefore conserved along isentropic surfaces (surfaces of constant q)

Forms of the First Law of Thermodynamics

$$dq = du + dw$$

$$dq = du + pd\alpha$$

$$dq = c_v dT + pd\alpha$$

$$dq = c_p dT - \alpha dp$$

$$dq = dh - \alpha dp$$

for a unit mass of gas

Some Definitions and Concepts

- **Adiabatic**: if a gas undergoes a change in its physical state (p, V, T) without any heat being added to it or being taken away from it, the change is said to be adiabatic

For adiabatic processes $dq=0$

$$\Rightarrow du = - dw$$

Definitions and Concepts (cont)

- **Air Parcel**

To gain some insights into vertical mixing and stability it is useful to consider the behavior of an infinitely small parcel of air that is assumed to be:

1. **thermally insulated** from its environment so that heat is not added to or taken away from the parcel (adiabatic)
2. at exactly the **same pressure as the environmental air** at the same level (the parcel immediately adjusts to the hydrostatic pressure at that level)
3. **moving slowly** enough so that its macroscopic kinetic energy is small (negligible fraction of the total energy)

3.8 Potential Temperature

- Consider an adiabatic process:

From the first law of thermodynamics :

$$dq = c_p dT - \alpha dp = 0 \quad (\text{for adiabatic processes } dq = 0)$$

Rearranging the Equation of State :

$$p\alpha = RT \Rightarrow \alpha = RT / p$$

Substituting for α :

$$c_p dT - \frac{RT}{p} dp = 0$$

$$\frac{c_p}{R} \frac{dT}{T} - \frac{dp}{p} = 0$$

$$\frac{c_p}{R} d \ln T - d \ln p = 0$$

$$\frac{c_p}{R} d \ln T - d \ln p = 0$$

Integrating from p_0 (typically 1000 mb) where by definition we let $T = \theta$ to p :

$$\frac{c_p}{R} \int_{\theta}^T d \ln T = \int_{p_0}^p d \ln p$$

$$\frac{c_p}{R} \ln \left(\frac{T}{\theta} \right) = \ln \left(\frac{p}{p_0} \right)$$

Taking antilogs:

$$\left(\frac{T}{\theta} \right)^{\frac{c_p}{R}} = \frac{p}{p_0}$$

$$\theta = T \left(\frac{p_0}{p} \right)^{R/c_p}$$

Poisson's Equation

For dry air, $R = R_d = 287 \text{ J K}^{-1} \text{ kg}^{-1}$ and $c_p = 1004 \text{ J K}^{-1} \text{ kg}^{-1}$,
therefore $R/c_p = 0.286$

- The potential temperature (θ) of an air parcel is the temperature that the air parcel would have if it were compressed or expanded adiabatically from its initial level (p, T) to a standard pressure level p_0 (usually taken to be 1000 mb)
- Potential temperature is **conserved** for adiabatic transformations
- Many atmospheric processes are close to adiabatic, so potential temperature is a useful parameter
- Potential temperature can be used as a tracer under adiabatic conditions

3.9 Dry Adiabatic Lapse Rate

- Interested in determining the rate of change of temperature with height of a dry parcel (remember our requirements for a parcel)
- For adiabatic motions, the potential temperature is conserved:

$$\Rightarrow \frac{d\theta}{dz} = 0$$

For a parcel undergoing adiabatic transformations :

$$\Rightarrow \frac{d\theta}{dz} = 0$$

But we showed that $\theta = T \left(\frac{p_0}{p} \right)^{R/c_p}$

Taking natural logs :

$$\ln \theta = \ln T + \frac{R}{c_p} (\ln p_0 - \ln p)$$

And then differentiating with respect to z :

$$\frac{1}{\theta} \frac{d\theta}{dz} = \frac{1}{T} \frac{dT}{dz} - \frac{R}{c_p} \frac{1}{p} \frac{dp}{dz} = 0$$

As the pressure of the parcel adjusts to the hydrostatic pressure at any level:

$$\frac{dp}{dz} = -\rho g$$

$$\Rightarrow \frac{1}{\theta} \frac{d\theta}{dz} = \frac{1}{T} \frac{dT}{dz} - \frac{R}{c_p p} \frac{dp}{dz} = \frac{1}{T} \frac{dT}{dz} + \frac{R\rho g}{c_p p} = 0$$

$$\Rightarrow \frac{dT}{dz} = - \frac{RT\rho}{p} \frac{g}{c_p}$$

But using the ideal gas law : $p = \rho RT$

$$\Rightarrow \frac{dT}{dz} = - \frac{g}{c_p} = -\Gamma_d$$

$$\frac{dT}{dz} = -\frac{g}{c_p} = -\Gamma_d$$

- Γ_d is referred to as the **dry adiabatic lapse rate**
- $\Gamma_d = g / c_p = 9.8^\circ\text{C} / \text{km} \approx 1^\circ\text{C} / 100\text{m}$
- Γ_d is the rate of change of temperature of a dry air parcel that is being raised or lowered adiabatically in the atmosphere

3.10 Water Vapor and Moisture Parameters

- We have seen that water vapor exerts pressure (vapor pressure) and affects the density of air
- The amount of water vapor in the air can be expressed in many different ways – we are going to look at these ways now
- We also need to consider what happens when phase changes occur

Mixing Ratio (w)

- The mixing ratio is the ratio of the mass of water vapor (m_v) to the mass of dry air (m_d) in a volume of air:

$$w = \frac{m_v}{m_d}$$

- w is usually expressed in g/kg
- r_v may be used in place of w to represent mixing ratio
- Typically observed values:
 - tropical marine air at the surface: ~ 20 g/kg
 - hot summer day in Colorado: ~ 6 g/kg
 - cold dry winter day in Colorado: ~ 0.1 g/kg

- The mixing ratio of an air parcel is conserved if neither condensation nor evaporation takes place
- As the volume of air in which the dry air and water vapor are contained is the same:

$$w = \frac{m_v}{m_d} = \frac{\rho_v}{\rho_d}$$

Specific Humidity (q)

- The ratio of the mass of water vapor (m_v) to the mass of air (dry air plus water vapor) is called the specific humidity:

$$q = \frac{m_v}{m_v + m_d}$$

- q and w are related as follows:

$$q = \frac{m_v}{m_v + m_d} = \frac{\frac{m_v}{m_d}}{\frac{m_v}{m_d} + 1} = \frac{w}{w + 1}$$

- As $w \ll 1 \Rightarrow q \approx w$

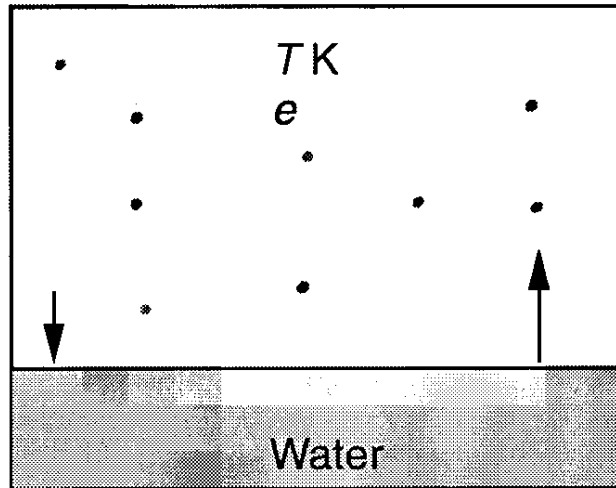
Relationship between Vapor Pressure and Mixing Ratio

- The partial pressure exerted by any constituent in a mixture of gases is proportional to the number of moles of the constituent in the mixture => the pressure e due to water vapor is therefore given by:

$$e = \frac{n_v}{n_v + n_d} p = \frac{\frac{m_v}{M_w}}{\frac{m_d}{M_d} + \frac{m_v}{M_w}} p = \frac{\frac{m_v}{M_d}}{\frac{m_d}{M_d} M_w + m_v} p = \frac{m_v}{m_d \varepsilon + m_v} p \quad \left(\varepsilon = \frac{R_d}{R_w} = \frac{M_w}{M_d} \right)$$

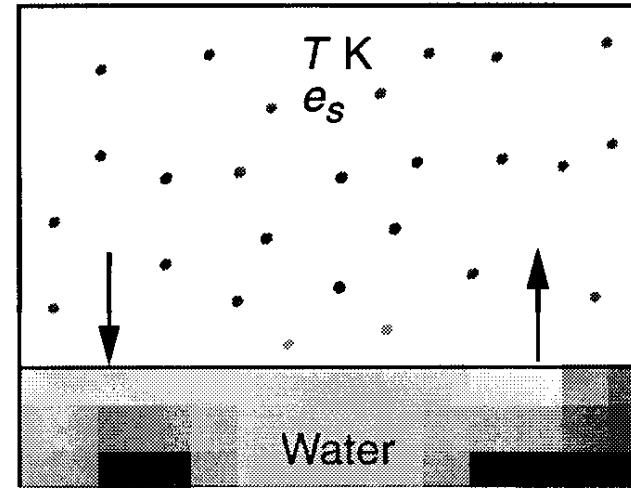
$$e = \frac{\frac{m_v}{m_d}}{\frac{m_v}{m_d} + \varepsilon} p = \frac{w}{w + \varepsilon} p$$

Saturation Vapor Pressure



(a) Unsaturated

When the rate of condensation is less than the rate of evaporation – air is unsaturated at temperature T



(b) Saturated

When water vapor pressure e increases to the point where the rate of condensation is equal to the rate of evaporation the air is saturated with respect to plane surface of pure water at temperature T .

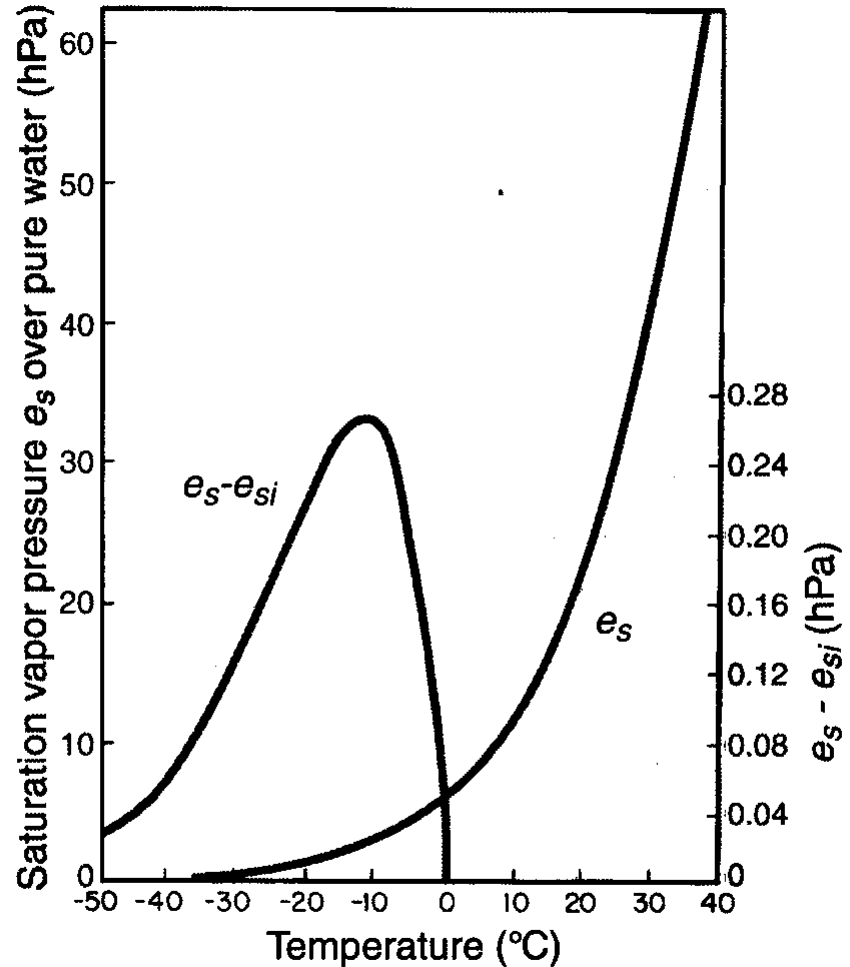
The pressure e_s exerted by the water vapor is called the saturation vapor pressure

- Similar arguments can be made for ice where e_{si} is the saturation vapor pressure for ice
- As the rate of evaporation from ice is less than that from water at any temperature:

$$e_s > e_{si}$$

- The rate at which water molecules evaporate from either water or ice increases with increasing temperature $\Rightarrow e_s$ and e_{si} increase with increasing temperature
- The magnitude of e_s and e_{si} depend only on temperature

- $e_s > e_{si}$ at all temperatures
- magnitude of $e_s - e_{si}$ reaches a peak value at -12°C
- ice particles in water-saturated air will grow due to the deposition of water vapor on them at the expense of water droplet growth



Saturation Mixing Ratios

- Saturation mixing ratio w_s is defined as the ratio of the mass of water vapor m_{vs} in a given volume of air that is saturated with respect to a plane surface of water to the mass m_d of dry air:

$$w_s = \frac{m_{vs}}{m_d}$$

$$w_s = \frac{m_{vs}}{m_d} = \frac{\rho'_{vs}}{\rho'_d} = \frac{\frac{e_s}{R_v T}}{\frac{p - e_s}{R_d T}} = \varepsilon \frac{e_s}{p - e_s} = 0.622 \frac{e_s}{p - e_s}$$

$$\Rightarrow w_s = \varepsilon \frac{e_s}{p - e_s}$$

For the temperatures of the Earth's atmosphere, $p \gg e_s$

$$\therefore w_s \cong \varepsilon \frac{e_s}{p} = 0.622 \frac{e_s}{p}$$

- At a given temperature, the saturation mixing ratio is inversely proportional to the total pressure
- As e_s depends only on temperature, w_s is therefore a function of temperature and pressure
- Lines of constant saturation mixing ratio are plotted on skew T-ln P chart as dashed green lines – apparent from these charts that at constant pressure w_s increases with increasing temperature, and at constant temperature w_s increases with decreasing pressure

Relative Humidity

- Relative humidity (RH):

$$RH = \frac{w}{w_s} 100$$

We showed earlier that $w \cong \frac{e}{p} \varepsilon$ and that $w_s \cong \frac{e_s}{p} \varepsilon$

$$\therefore \frac{w}{w_s} \cong \frac{e}{e_s}$$

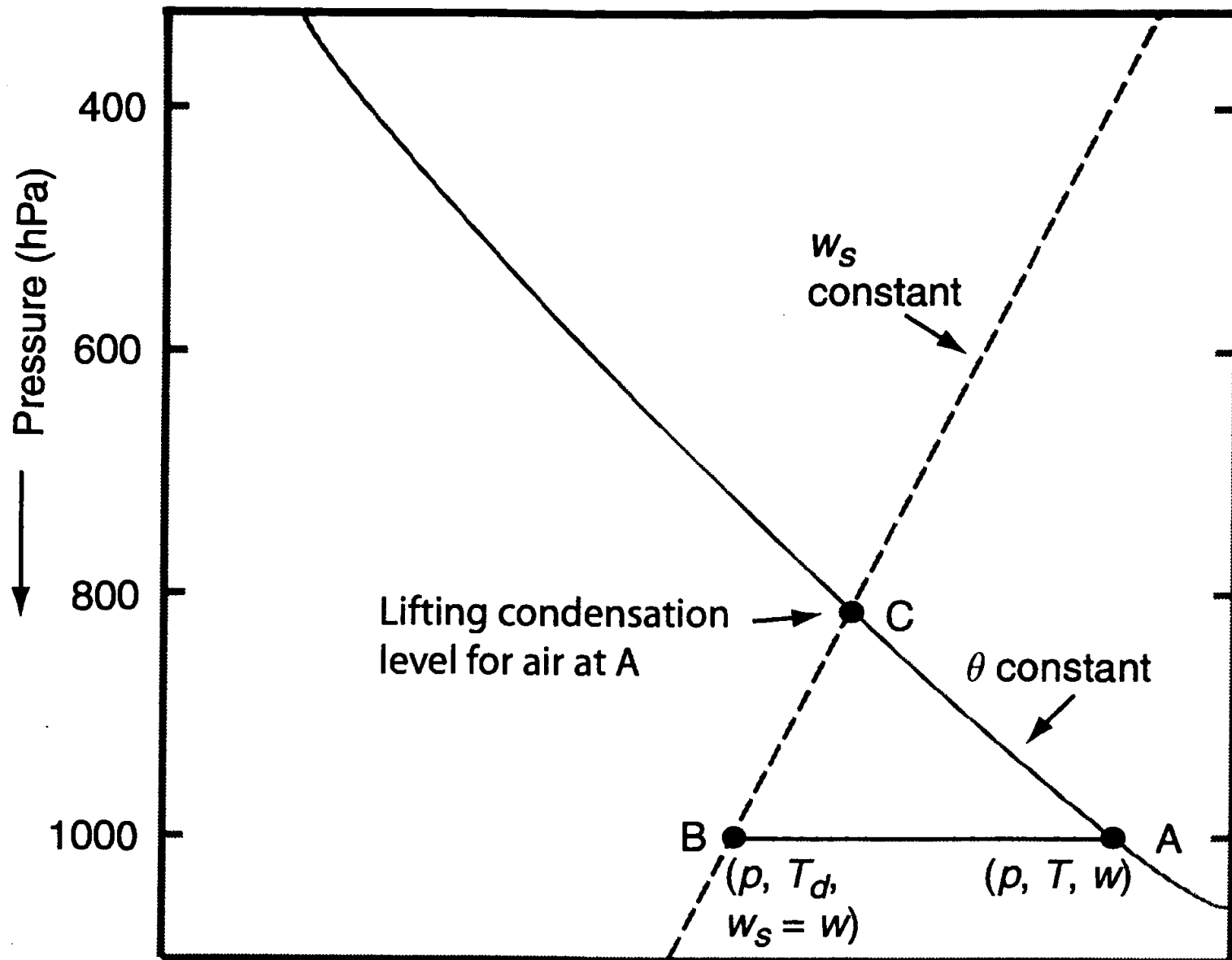
$$\therefore RH = \frac{w}{w_s} 100 = \frac{e}{e_s} 100$$

Dew Point and Frost Point

- **Dew point temperature T_d** : the temperature to which air must be cooled at constant pressure for it to become saturated with respect to a plane surface of pure water => the dew point is therefore the temperature at which the actual mixing ratio w becomes equal to the saturation mixing ratio w_s
- **Frost Point**: temperature to which air must be cooled at constant pressure to saturate it with respect to a plane surface of pure ice
- At the earth's surface the pressure varies little in time and space so the dew point is a good indicator of the moisture content of the air
- Relative humidity depends upon both the temperature and the moisture of the air and as such is not a good indicator of the moisture content of the air (RH may drop as much as 50% on a sunny day from morning to afternoon just because of the rise in temperature)

Lifting Condensation Level

- LCL: the level to which an unsaturated (but moist) parcel of air can be lifted adiabatically before it becomes saturated
- During lifting the mixing ratio w and the potential temperature of the air remain constant, but the saturation mixing ratio w_s decreases until it becomes equal to w at the LCL
- LCL is located at the intersection of the potential temperature line passing through the temperature T and pressure p of the parcel of air and the w_s line that passes through the pressure p and dew point T_d of the air parcel
=> knowledge of either the dew point or the LCL are sufficient to determine the other



Wet-Bulb Temperature (T_w)

- Wet-bulb temperature: temperature to which a parcel of air is cooled by evaporation of water into it at constant pressure until the air is saturated with respect to water
- Measured with a thermometer in which the glass bulb is covered with a moist cloth over which ambient air is drawn

- T_w and T_d are different: if the unsaturated air approaching the wet bulb has a mixing ratio of w , then T_d is the temperature to which the air must be cooled at constant pressure to become saturated. The evaporation from the wet bulb adds a certain amount of water vapor to the air and the air that leaves the wet bulb then has a mixing ratio of w' that saturates at temperature T_w . If the air approaching the wet bulb is unsaturated, w' is greater than w

$\Rightarrow$

$$T_d \leq T_w \leq T$$

where the equality signs apply only to saturated air

3.11 Saturated-Adiabatic and Pseudoadiabatic Processes

- **Saturated adiabatic**: as a parcel rises it cools dry adiabatically until the parcel becomes saturated. Further lifting will result in condensation and the release of latent heat. If we retain all the condensate in the parcel, the process can be considered reversible as evaporation of the condensate during the descent of the parcel will consume the latent heat freed during the ascent. Such a process is called saturated-adiabatic
- **Pseudoadiabatic**: if the condensate drops out of the parcel during ascent, the process is irreversible as the condensate is not available for evaporation on the descent. This process is called pseudoadiabatic.
- The amount of heat carried by the amount of condensation products is small compared to that carried by the air itself, and the saturated-adiabatic and pseudoadiabatic rates are therefore similar

Derivation of Saturated Adiabatic Lapse Rate

- As an air parcel is lifted it cools dry adiabatically until it becomes saturated. Further ascent results in condensation. Latent heat is released and the parcel cools at a lesser rate than the dry adiabatic lapse rate

From the first law of thermodynamics :

$$dq = c_p dT - \alpha dp = c_p dT + g dz$$

If the saturation mixing ratio of the air with respect to water is w_s , the quantity of heat dq released into (or absorbed from) a unit mass of dry air due to condensation (or evaporation) of liquid water is $-L_v dw_s$ where L_v is the latent heat of condensation. Therefore :

$$-L_v dw_s = c_p dT + g dz$$

$$-L_v dw_s = c_p dT + g dz$$

Dividing by $c_p dz$ and rearranging :

$$\frac{dT}{dz} = -\frac{L_v}{c_p} \frac{dw_s}{dz} - \frac{g}{c_p}$$

Now making use of the chain rule $\frac{dw_s}{dz}$ can be written as :

$$\frac{dw_s}{dz} = \frac{dw_s}{dT} \frac{dT}{dz}$$

Substituting this into the above equation:

$$\frac{dT}{dz} = -\frac{L_v}{c_p} \frac{dw_s}{dT} \frac{dT}{dz} - \frac{g}{c_p} = -\frac{dT}{dz} \left(\frac{L_v}{c_p} \frac{dw_s}{dT} + \frac{\frac{g}{c_p}}{\frac{dT}{dz}} \right)$$

Rearranging this expression we get :

$$1 + \frac{L_v}{c_p} \frac{dw_s}{dT} = - \frac{\frac{g}{c_p}}{\frac{dT}{dz}}$$

$$\frac{dT}{dz} = - \frac{\frac{g}{c_p}}{1 + \frac{L_v}{c_p} \frac{dw_s}{dT}}$$

$$\Gamma_s = - \frac{dT}{dz} = \frac{\Gamma_d}{1 + \frac{L_v}{c_p} \frac{dw_s}{dT}}$$

$$\Gamma_s = -\frac{dT}{dz} = \frac{\Gamma_d}{1 + \frac{L_v}{c_p} \frac{dw_s}{dT}}$$

- If condensate does not fall out of the parcel then Γ_s is called the saturated-adiabatic lapse rate
- If condensate falls out of the parcel then Γ_s is called the pseudoadiabatic lapse rate
- Since dw_s/dT is always positive $\Rightarrow \Gamma_s < \Gamma_d$
- Values of Γ_s range from 4 K km⁻¹ near the ground in warm, humid air where dw_s/dT is very large to more typical values of 6-7 K km⁻¹ in the middle troposphere. Near the tropopause Γ_s is only slightly less than Γ_d as the moisture capacity at these temperatures is small
- Lines of Γ_s on thermodynamic diagrams are referred to as saturated adiabats or pseudoadiabats

3.12 Equivalent Potential Temperature

Using the first law of thermodynamics and the equation of state :

$$dq = c_p dT - \alpha dp = c_p dT - \frac{RT}{p} dp$$

$$\Rightarrow \frac{dq}{T} = c_p \frac{dT}{T} - R \frac{dp}{p}$$

$$\text{Now } \theta = T \left(\frac{p_0}{p} \right)^{R/c_p} \Rightarrow \ln \theta = \ln T - \frac{R}{c_p} \ln p + \text{const}$$

Differentiating :

$$c_p \frac{d\theta}{\theta} = c_p \frac{dT}{T} - R \frac{dp}{p}$$

$$\Rightarrow \frac{dq}{T} = c_p \frac{d\theta}{\theta}$$

$$\text{Now } dq = -L_v dw_s$$

$$\therefore \frac{-L_v dw_s}{c_p T} = \frac{d\theta}{\theta} = -d\left(\frac{L_v w_s}{c_p T}\right) + w_s d\left(\frac{L_v}{c_p T}\right)$$

Now $w_s d\left(\frac{L_v}{c_p T}\right)$ can be shown to be small so :

$$d\ln\theta \cong -d\left(\frac{L_v w_s}{c_p T}\right)$$

By requiring that at low temperatures as $w_s/T \rightarrow 0, \theta \rightarrow \theta_e$:

$$\int_{\theta}^{\theta_e} d\ln\theta = -\frac{L_v}{c_p} \int_{\frac{w_s}{T}}^0 d\left(\frac{w_s}{T}\right) \Rightarrow -\frac{L_v}{c_p} \frac{w_s}{T} = \ln\left(\frac{\theta}{\theta_e}\right)$$

$$\theta_e = \theta \exp\left(\frac{L_v w_s}{c_p T}\right)$$

$$\theta_e = \theta \exp \left(\frac{L_v}{c_p} \frac{w_s}{T} \right)$$

- θ_e is called the equivalent potential temperature
- θ_e is the potential temperature of a parcel that has been lifted so that its saturation mixing ratio is zero
- To get θ_e on a thermodynamic diagram lift the parcel pseudoadiabatically until the pseudoadiabat parallels the dry adiabat after which compress dry adiabatically to 1000mb
- θ_e is conserved during both dry and saturated-adiabatic processes and is a valuable tracer of air masses

3.13 Static Stability

The vertical equation of motion is given by :

$$\frac{dw}{dt} = -\frac{1}{\rho} \frac{dp}{dz} - g$$

so vertical motion is due to the imbalance between the pressure gradient force and the acceleration due to gravity

Now if we assume that the environmental air is in hydrostatic balance :

$$\frac{dp}{dz_{env}} = -\rho_{env} g$$

But for an air parcel

$$\frac{dw}{dt} = -\frac{1}{\rho_{par}} \frac{dp}{dz_{par}} - g$$

Now if we assume that the vertical pressure gradient acting on a parcel is the same as the environmental vertical pressure

gradient at that level: $\frac{dp}{dz_{par}} = \frac{dp}{dz_{env}} = \frac{dp}{dz}$ then

$$\begin{aligned}\frac{dw}{dt} &= -\frac{1}{\rho_{\text{par}}} \frac{dp}{dz_{\text{env}}} - g = -\frac{1}{\rho_{\text{par}}} (-\rho_{\text{env}} g) - g \\ &= \frac{\rho_{\text{env}} g}{\rho_{\text{par}}} - g = g \left(\frac{\rho_{\text{env}}}{\rho_{\text{par}}} - 1 \right)\end{aligned}$$

$$\therefore \frac{dw}{dt} = g \left[\frac{\rho_{\text{env}} - \rho_{\text{par}}}{\rho_{\text{par}}} \right]$$

• If $\rho_{\text{env}} > \rho_{\text{par}} \Rightarrow dw/dt > 0$

$\rho_{\text{env}} = \rho_{\text{par}} \Rightarrow dw/dt = 0$ neutral

$\rho_{\text{env}} < \rho_{\text{par}} \Rightarrow dw/dt < 0$

• Therefore a parcel starting at rest will accelerate upward if it is less dense than the surrounding air. If $\rho_{\text{env}} = \rho_{\text{par}}$ a parcel at rest will stay at rest and a parcel in motion will continue to move at constant speed

Static Stability in terms of Temperature

$$\text{Using } p = \rho RT \Rightarrow \rho_{\text{env}} = \frac{p}{RT_{\text{env}}} \text{ and } \rho_{\text{par}} = \frac{p}{RT_{\text{par}}}$$

$$\frac{dw}{dT} = g \left[\frac{\rho_{\text{env}} - \rho_{\text{par}}}{\rho_{\text{par}}} \right] = g \left[\frac{\frac{p}{RT_{\text{env}}} - \frac{p}{RT_{\text{par}}}}{\frac{p}{RT_{\text{par}}}} \right] = g \left[\frac{\frac{1}{T_{\text{env}}} - \frac{1}{T_{\text{par}}}}{\frac{1}{T_{\text{par}}}} \right]$$

$$\therefore \frac{dw}{dt} = g \left[\frac{T_{\text{par}} - T_{\text{env}}}{T_{\text{env}}} \right]$$

$$\text{Similarly: } \frac{dw}{dt} = g \left[\frac{\theta_{\text{par}} - \theta_{\text{env}}}{\theta_{\text{env}}} \right]$$

- If $T_{\text{par}} > T_{\text{env}} \Rightarrow dw/dt > 0$

$$T_{\text{par}} = T_{\text{env}} \Rightarrow dw/dt = 0$$

$$T_{\text{par}} < T_{\text{env}} \Rightarrow dw/dt < 0$$

Therefore a parcel starting at rest will accelerate upward if it is warmer than the surrounding air

Static Stability in terms of Γ and Γ_d

Making use of a Taylor series expansion :

$$T_{\text{env}} = T_0 + \frac{dT}{dz_{\text{env}}} \delta z + \dots$$

$$T_{\text{par}} = T_0 + \frac{dT}{dz_{\text{par}}} \delta z + \dots$$

where T_0 is the temperature at the level at which the parcel originated

If δz is small then :

$$T_{\text{env}} \cong T_0 - \Gamma \delta z$$

$$T_{\text{par}} \cong T_0 - \Gamma_d \delta z$$

where $\Gamma = -\frac{dT}{dz_{\text{env}}}$ (env lapse rate) and $\Gamma_d = -\frac{dT}{dz_{\text{par}}}$ (dry adiab lapse rate)

But we showed that :

$$\begin{aligned}
 \frac{dw}{dt} &= g \left[\frac{T_{\text{par}} - T_{\text{env}}}{T_{\text{env}}} \right] \approx g \left[\frac{(T_0 - \Gamma_d \delta z) - (T_0 - \Gamma \delta z)}{T_{\text{env}}} \right] \\
 &= g \left[\frac{(-\Gamma_d + \Gamma) \delta z}{T_{\text{env}}} \right] \\
 \therefore \frac{dw}{dt} &= g \left[\frac{\Gamma - \Gamma_d}{T_{\text{env}}} \right] \delta z
 \end{aligned}$$

• If $\Gamma_d > \Gamma \Rightarrow dw/dt < 0$

$\Gamma_d = \Gamma \Rightarrow dw/dt = 0$

$\Gamma_d < \Gamma \Rightarrow dw/dt > 0$

Static Stability in terms of θ

$$\theta = T \left(\frac{p_0}{p} \right)^{R/c_p} \Rightarrow \ln \theta = \ln T + \frac{R}{c_p} (\ln p_0 - \ln p)$$

$$\frac{1}{\theta} \frac{d\theta}{dz} = \frac{1}{T} \frac{dT}{dz} - \frac{R}{c_p p} \frac{dp}{dz} = \frac{1}{T} \frac{dT}{dz} - \frac{R}{c_p} \frac{1}{\rho R T} (-\rho g)$$

$$\frac{1}{\theta} \frac{d\theta}{dz} = \frac{1}{T} \frac{dT}{dz} + \frac{g}{c_p T} = \frac{1}{T} \left[\frac{dT}{dz} + \frac{g}{c_p} \right] = \frac{1}{T} \left[\frac{g}{c_p} - \left(-\frac{dT}{dz} \right) \right]$$

$$\therefore \frac{T}{\theta} \frac{d\theta}{dz} = \Gamma_d - \Gamma$$

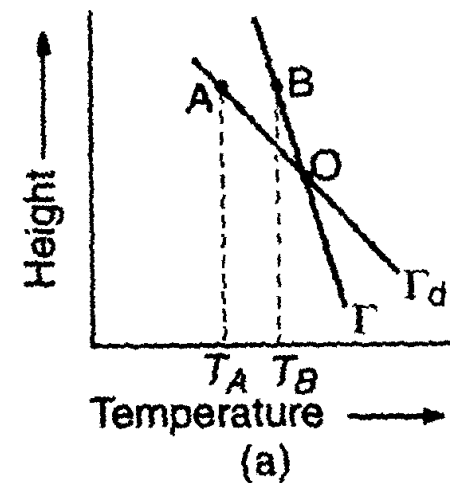
- When $\Gamma_d > \Gamma \Rightarrow d\theta/dz > 0$ and $dw/dt < 0$ stable

$$\Gamma_d = \Gamma \Rightarrow d\theta/dz = 0 \text{ and } dw/dt = 0 \quad \text{neutral}$$

$$\Gamma_d < \Gamma \Rightarrow d\theta/dz < 0 \text{ and } dw/dt > 0 \quad \text{unstable}$$

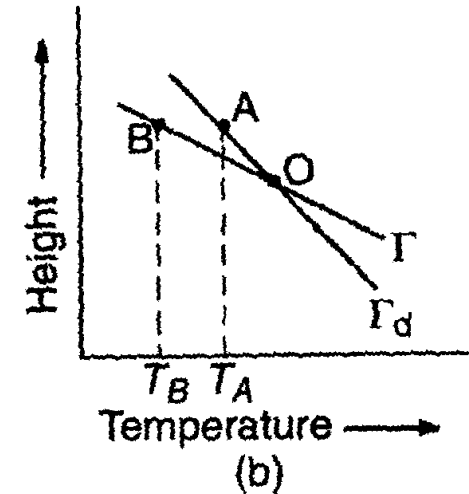
- Similar results may be obtained for saturated air (using Γ_s and $d\theta_e/dz$)

- $\Gamma < \Gamma_d$
- If a parcel is forced to rise to the level of points A and B then the parcel temperature will fall to T_A which is lower than the ambient temperature T_B



- As the parcel pressure immediately adjusts to the pressure of the environment, it is clear from the ideal gas equation that the density of the parcel is greater than that of the environmental air – the parcel will therefore return to its original level
- If the parcel is displaced downwards from O it becomes warmer than the ambient air and if left to itself will tend to rise back to O
- In both cases the air parcel encounters a restoring force after being displaced which inhibits vertical mixing (the greater the difference $\Gamma_d - \Gamma$ the greater the restoring force and the greater the static stability)
- $\Gamma < \Gamma_d$ corresponds to stable stratification (or positive static stability) for unsaturated air parcels
- Layers of air with negative lapse rates (temperature increases with height) are called inversions – marked by very strong static stability

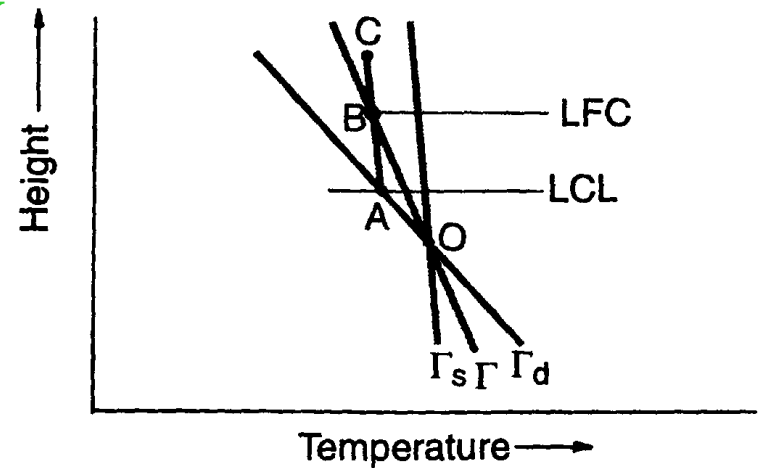
- $\Gamma > \Gamma_d$
- A parcel of unsaturated air displaced from O will arrive at A with a temperature greater than that of the environment – it will therefore be less dense than the environmental air, and left to itself, it will continue to rise



- Similarly if the parcel is displaced downward it will be cooler than the ambient air and will continue to sink if left to itself
- Such unstable conditions generally do not exist in the atmosphere for long as the instability is reduced by strong vertical mixing. An exception to this is in the layer just above the ground when there is strong heating from below.
- Similar arguments may be made for saturated air – saturated parcels will be stable, neutral or unstable with respect to vertical displacements depending on whether $\Gamma < \Gamma_s$, $\Gamma = \Gamma_s$, or $\Gamma > \Gamma_s$ respectively

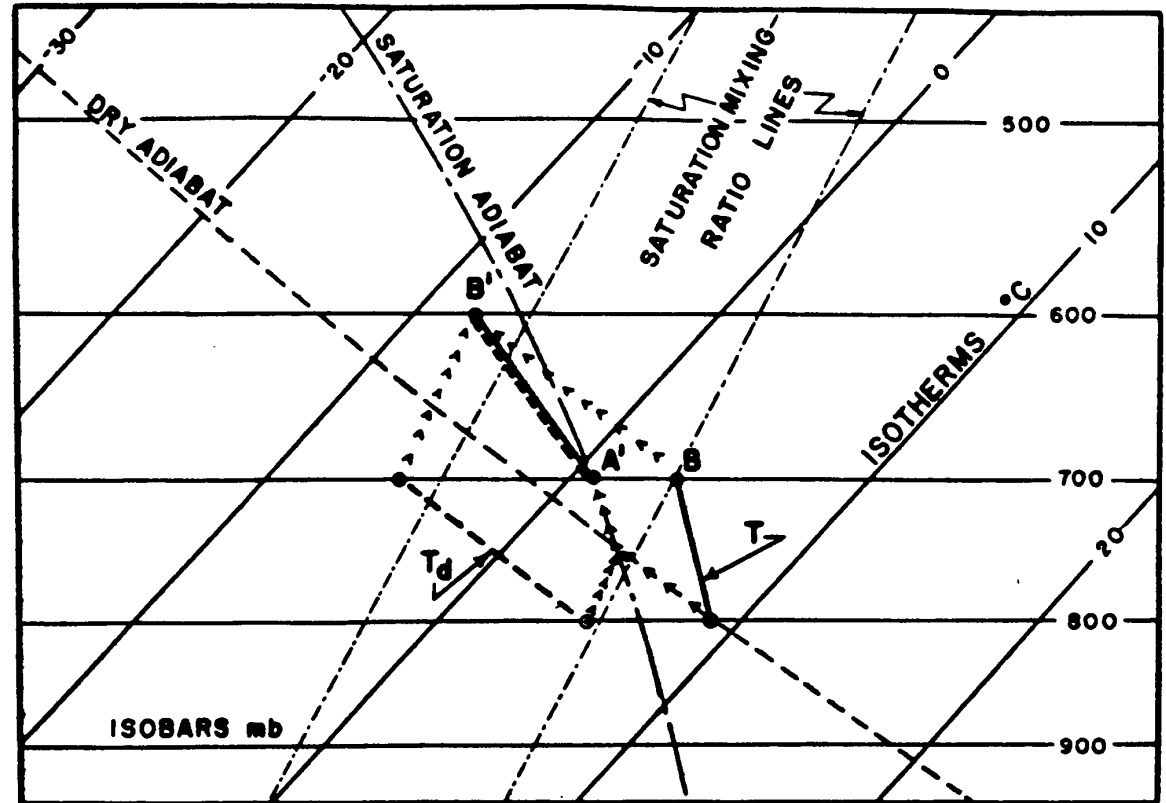
Conditional Instability

- $\Gamma_d > \Gamma > \Gamma_s$
- Parcel lifted from its equilibrium level O will cool adiabatically until it reaches its LCL at A – at this level the parcel is colder and denser than the ambient air
- Further lifting will cause the parcel to cool at the moist adiabatic lapse rate
- If the parcel is sufficiently moist, the moist adiabat through A will cross the environmental temperature sounding at B
- Up to B, energy is required to force the parcel above its equilibrium level O
- Above B the parcel is now warmer than the environment and this positive buoyancy results in the upward motion of the parcel even without the forced lifting
- B is called the level of free convection (LFC) – the LFC is dependent on the amount of moisture in the parcel as well as the magnitude of the lapse rate Γ
- When $\Gamma_d > \Gamma > \Gamma_s$ the atmosphere is said to be conditionally unstable



Convective or Potential Instability

- The unsaturated layer AB between 700 and 800 mb is stable
- If the layer were lifted 100mb its new lapse rate would be A'B'
- The layer is now saturated and its stability is then found by comparing the slope of A'B' to the saturation adiabats – the layer is now unstable
- The original layer AB is potentially unstable, the instability being releasable by lifting of 100mb or more



Thermodynamic Stability	Lapse Rate Relationship	Potential Temperature Relationship	Parcel Behaviour
Absolutely stable	$\Gamma < \Gamma_s$	$d\theta_e/dz > 0$	Parcel will move back to original position regardless of whether or not saturation occurs
Conditionally stable	$\Gamma_s \leq \Gamma < \Gamma_d$	$d\theta_e/dz < 0$ but $d\theta/dz > 0$	Parcel accelerates away from original position if displaced and saturated. If not saturated parcel will return to original position
Absolutely unstable	$\Gamma \geq \Gamma_d$	$d\theta/dz \leq 0$	Parcel will accelerate away from its original position when displaced