

## Part 4. Atmospheric Dynamics

We apply Newton's Second Law:

$$ma = \Sigma F_i$$

to the atmosphere.

In Cartesian coordinates



$$u = \frac{dx}{dt}$$

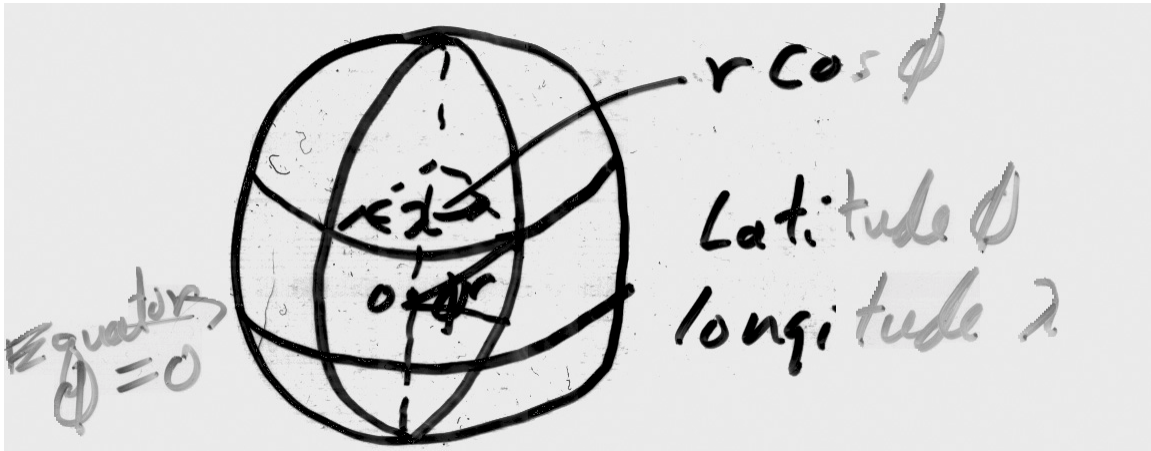
$$v = \frac{dy}{dt}$$

$$w = \frac{dz}{dt}$$

$$a_i = \frac{1}{m} \Sigma F_i$$

$$a_x = \frac{du}{dt}; a_y = \frac{dv}{dt}; a_z = \frac{dw}{dt}$$

## Coordinate Systems



$x$  = distance east of Greenwich meridian

$y$  = distance north of equator

$r$  = distance from center of earth

Then,

$$dx = r d\lambda \cos \phi; dy = r d\phi$$

$$R_E = \text{mean radius of earth} = 6.37 \times 10^6 \text{ km}$$

$$z = \frac{\phi - \phi_0}{g_0}$$

$\phi - \phi_0$  = mean sea level geopotential

$g_0$  = sea level gravity

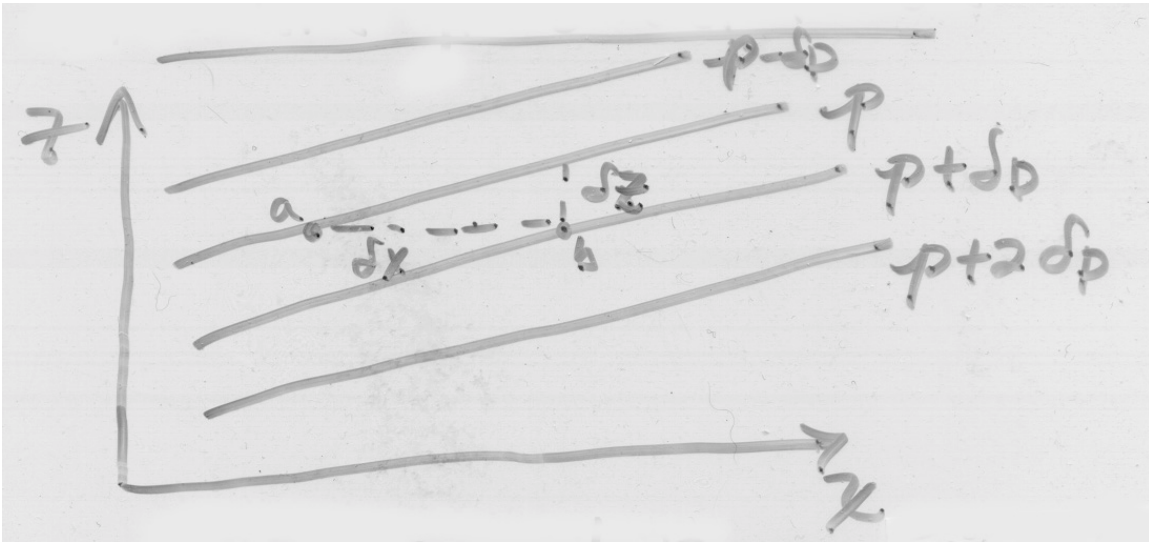
## Velocity Components

$$u = \frac{dx}{dt} = R_E \cos \phi \frac{d\lambda}{dt} \quad (\text{zonal velocity component})$$

$$v = \frac{dy}{dt} = R_E \frac{d\phi}{dt} \quad (\text{meridional component})$$

$$w = \frac{dz}{dt} \quad \square \quad \frac{dr}{dt}$$

## Forces



$$p_x = -\frac{1}{\rho} \frac{\partial p}{\partial x}; p_z = -\frac{1}{\rho} \frac{\partial p}{\partial z}; p_y = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

## Friction

Friction represents the collective effects of all scales of motion smaller than scales under consideration. Friction has its greatest impact near the earth's surface.

A simple model:

$$\left. \begin{aligned} F_x &= -C_D u^2 \\ F_y &= -C_D v^2 \end{aligned} \right\} \begin{array}{l} \text{slows} \\ \text{wind} \end{array}$$

## Pressure as a Vertical Coordinate

Why use  $p$ ?

- Large scale motions are hydrostatic so  $p$  monotonically decreases with height.
- $P$ -surfaces are nearly horizontal and are useful in analysis.

Vertical velocity:  $\omega = \frac{dp}{dt}$

$$\omega > 0 \Downarrow; \omega < 0 \Uparrow$$

$$\omega = \frac{dp}{dt} \approx -\rho g_0 W$$

## Other Vertical Coordinates

$\sigma = p / p_s$ ; Advantage is lower boundary  $\sigma = 1$  is tropographic surface.

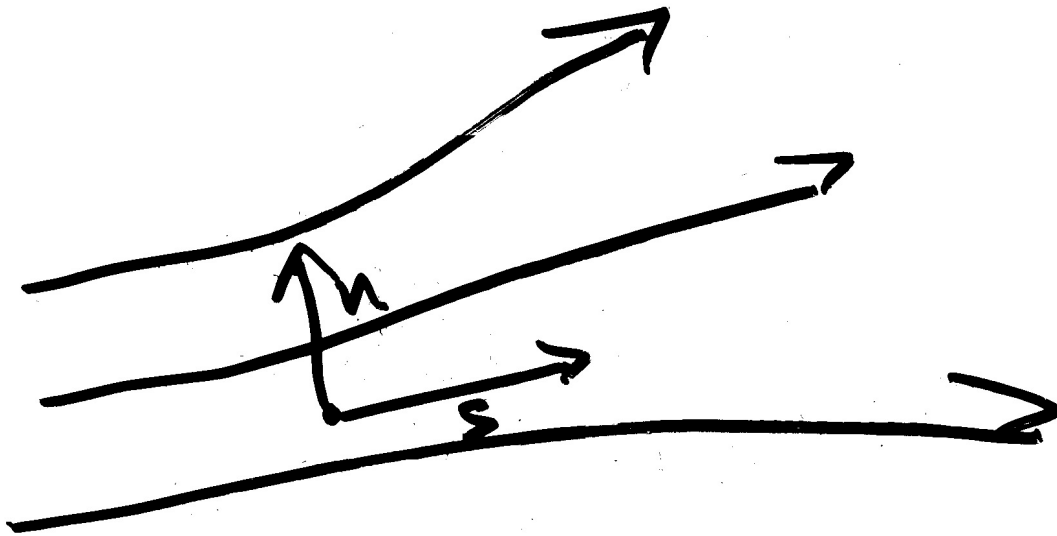
$\theta$ : (isentropic coordinate) horizontal motions tend to follow isentropic surfaces.

Also terrain-following:

$$\hat{z} = \left( \frac{z - z_s}{H - z_s} \right) H$$

Where  $z_s$  is height of topography, and  $H$  is the depth of model atmosphere.

## Natural Coordinates



Defined with respect to stream lines.

$$v = \frac{ds}{dt} \text{ and } \frac{dn}{dt} = 0$$

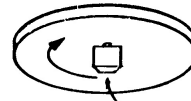
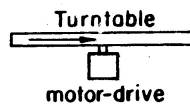
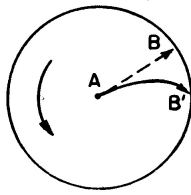
## Apparent Forces

The coordinate system we are used to rotates with an angular velocity

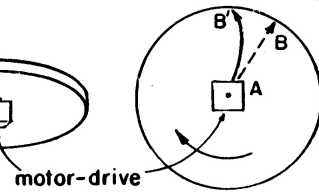
$$\Omega = 2\pi \text{ rad day}^{-1} = 7.292 \times 10^{-5} \text{ s}^{-1}$$

The effects of the rotation create “apparent” forces that are due solely to the fact that the coordinate system rotates.

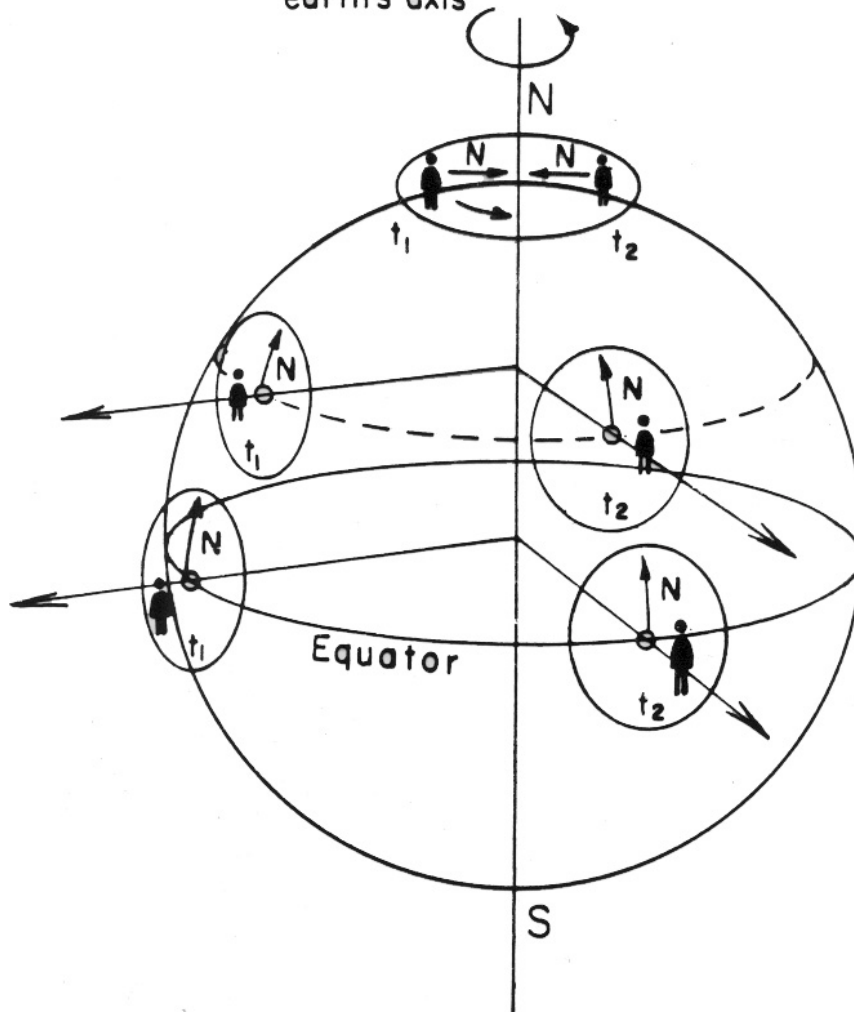
Top View

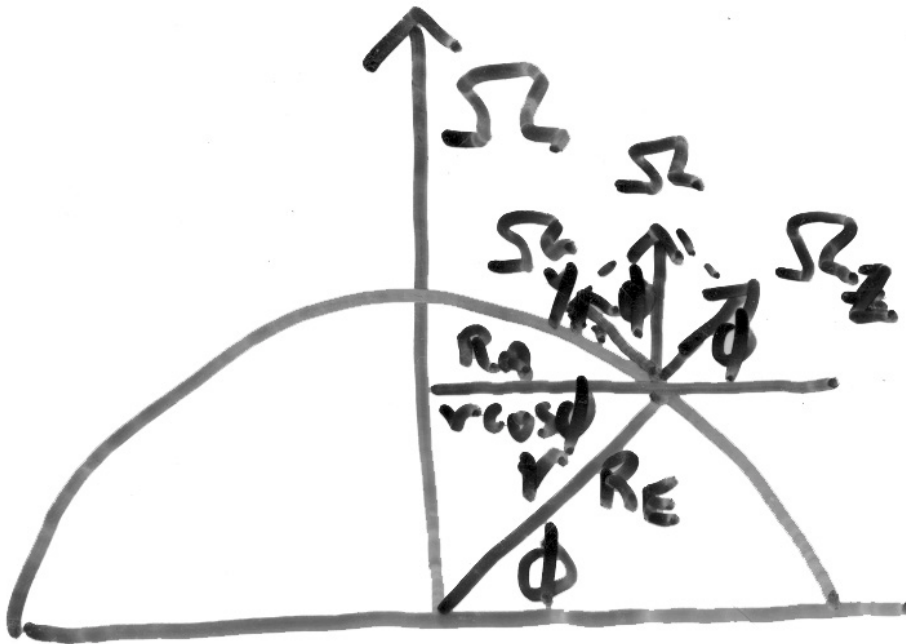


Bottom View



Sense of rotation of  
earth's axis





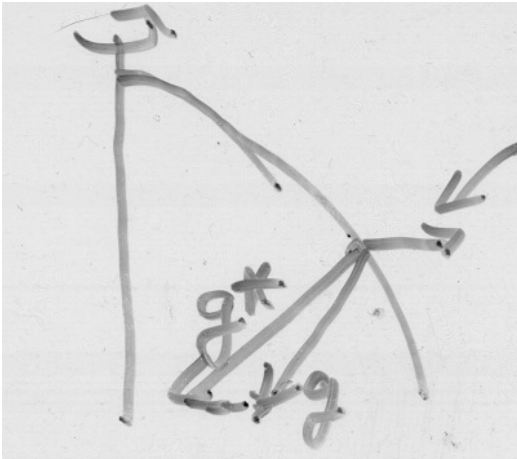
Earth's rotation vector ( $\vec{\Omega}$ ) can be resolved into two comp's, a radial component and a tangential component.

The linear velocity of a point fixed on the earth is

$$u = \Omega R_E \cos \phi.$$

## Effective Gravity

The force per unit mass called gravity or effective gravity  $g$  is the vector sum of the true gravitational attraction  $g^*$  that draws objects to the center of mass of earth and apparent centrifugal force that pulls objects outward from the axes of rotation with a force  $\Omega^2 R_A = \Omega^2 R_E \cos \phi$ .



$$\Omega^2 R_A = \Omega^2 R_E \cos \phi$$

## Coriolis Force (Farce?)

As a parcel moves toward or away from the axis of rotation its angular momentum is conserved:

$$\frac{d}{dt}(\Omega R_A^2 + \omega R_A^2) = 0$$

Where  $\omega$  is the relative angular velocity due to an air parcel moving at the surface of the earth, hence

$$\omega = u / R_A.$$

Thus,

$$\frac{d}{dt}(\Omega R_A^2 + u R_A) = 0.$$

Differentiating we find:

$$2\Omega R_A \dot{R}_A + u \dot{R}_A + \dot{R}_A \dot{u} = 0$$

or

$$\frac{du}{dt} = -\left(2\Omega + \frac{u}{R_A}\right) \dot{R}_A.$$

But

$$\dot{R}_A = \frac{d}{dt}(R_E \cos \phi)$$

$$\dot{R}_A = -R_E \sin \phi \frac{d\phi}{dt}$$

$R_E \frac{d\phi}{dt}$  is the linear velocity on a meridian circle or

$$R_E \frac{d\phi}{dt} = \frac{dy}{dt} = v.$$

thus,  $\dot{R}_A = -v \sin \phi$  and

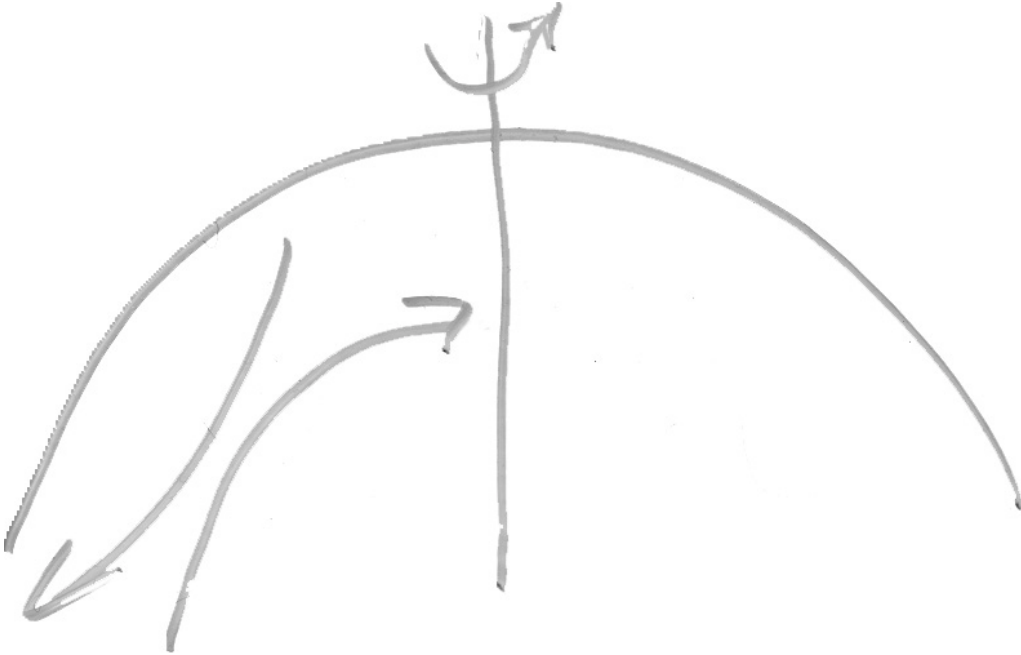
$$\frac{du}{dt} = + \left( 2\Omega + \frac{u}{R_A} \right) v \sin \phi$$

or

$$\frac{du}{dt} = 2\Omega v \sin \phi + \frac{uv}{R_A} \sin \phi$$

Except near poles the first term dominates, or

$$\boxed{\frac{du}{dt} \approx 2\Omega v \sin \phi.}$$



A northward moving parcel will be turned to the east and a southward to the west.

For a parcel moving along a latitude circle, the parcel experiences a relative acceleration

$$a_y = u^2 / R_A$$

and in an absolute reference frame it experiences an acceleration

$$a = (U + u)^2 / R_A.$$

But,

$$\text{relative}(a_y) = \text{absolute}(a_y) + \text{apparent}(a_y)$$

or

$$\frac{u^2}{R_A} = \left( \frac{U + u}{R_A} \right)^2 + \text{apparent}(a_y)$$

$$\frac{u^2}{R_A} = \frac{U^2}{R_A} + \frac{2Uu}{R_A} + \frac{u^2}{R_A} + \text{apparent}(a_y)$$

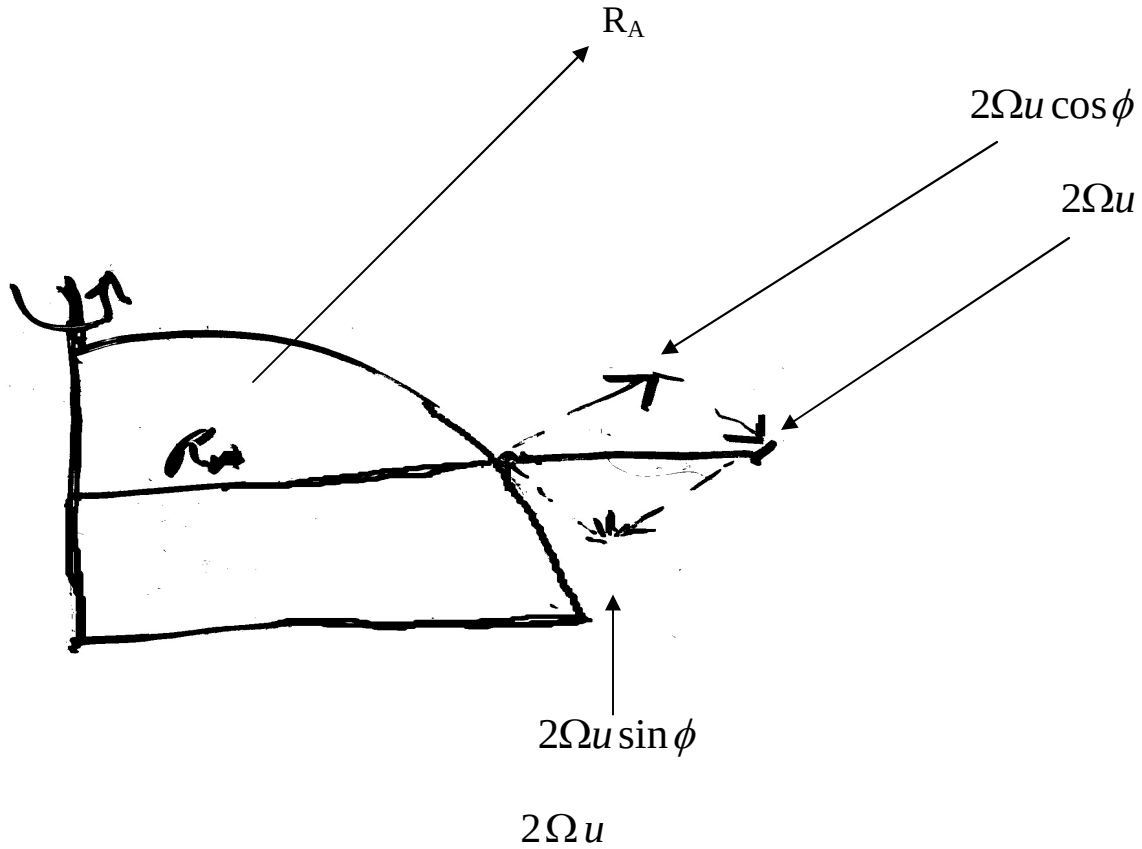
Let  $U = \Omega R_A$ .

$$\text{Apparent}(a_y) = -2\Omega u - \Omega^2 R_A.$$

Coriolis force due to u-motion      What's added to form effective gravity  $g$ .

Since second term is incorporated in  $g$ ,

$$a_y = -2\Omega u.$$



Thus

$$\frac{dv}{dt} = -2\Omega u \sin \phi$$

### Horizontal Equations of Motion

$$\frac{du}{dt} = -\alpha \frac{\partial p}{\partial x} + 2\Omega v \sin \phi + F_x$$

$$\frac{dv}{dt} = -\alpha \frac{\partial p}{\partial x} - 2\Omega u \sin \phi + F_y$$

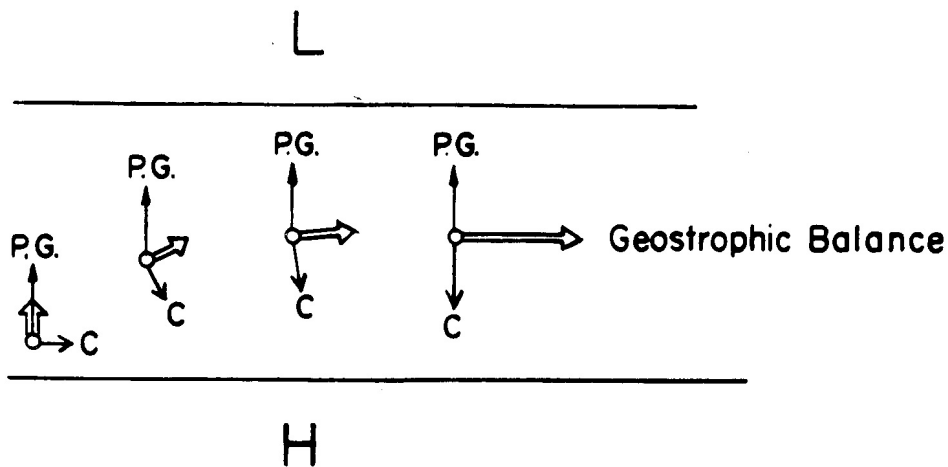
Above the atmospheric boundary layer (ABL), friction is unimportant, the air flow approaches equilibrium, such that

$$\frac{du}{dt} = 0 = \frac{dv}{dt}$$

or

$$\begin{aligned} \alpha \frac{\partial p}{\partial x} &= fv \\ \alpha \frac{\partial p}{\partial y} &= -fu \end{aligned}$$

Called geostrophic equilibrium.



At a given latitude, for large  $\frac{\partial p}{\partial n}$ ,  $V_g$  is large. At low latitudes,  $f \rightarrow 0$ ,

$V_g$  must be larger for a given pressure gradient in order to maintain geostrophic flow. Geostrophic balance is rarely achieved at low latitudes.

It can be readily shown:

$$dp = \frac{\partial p}{\partial n} dn + \frac{\partial p}{\partial z} dz.$$

On a constant pressure surface

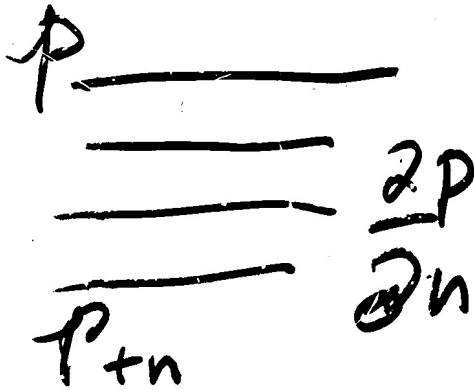
$$dp = 0.$$

Thus,

$$\frac{\partial p}{\partial n} = - \overbrace{\frac{\partial p}{\partial t}}^{g\rho} \left( \frac{dz}{dn} \right)_p$$

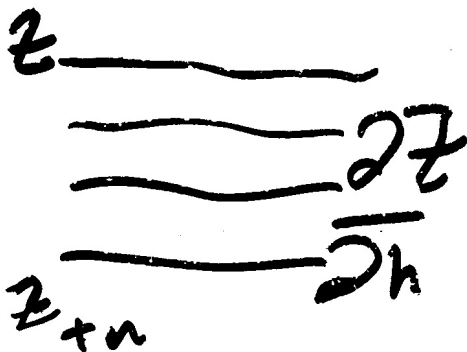
or

$$\frac{1}{\rho} \frac{\partial p}{\partial n} = g \left( \frac{dz}{dn} \right)_p$$



or in terms of geopotential heights:

$$-\frac{1}{\rho} \frac{\partial p}{\partial n} = -g_0 \frac{\partial Z}{\partial n}$$

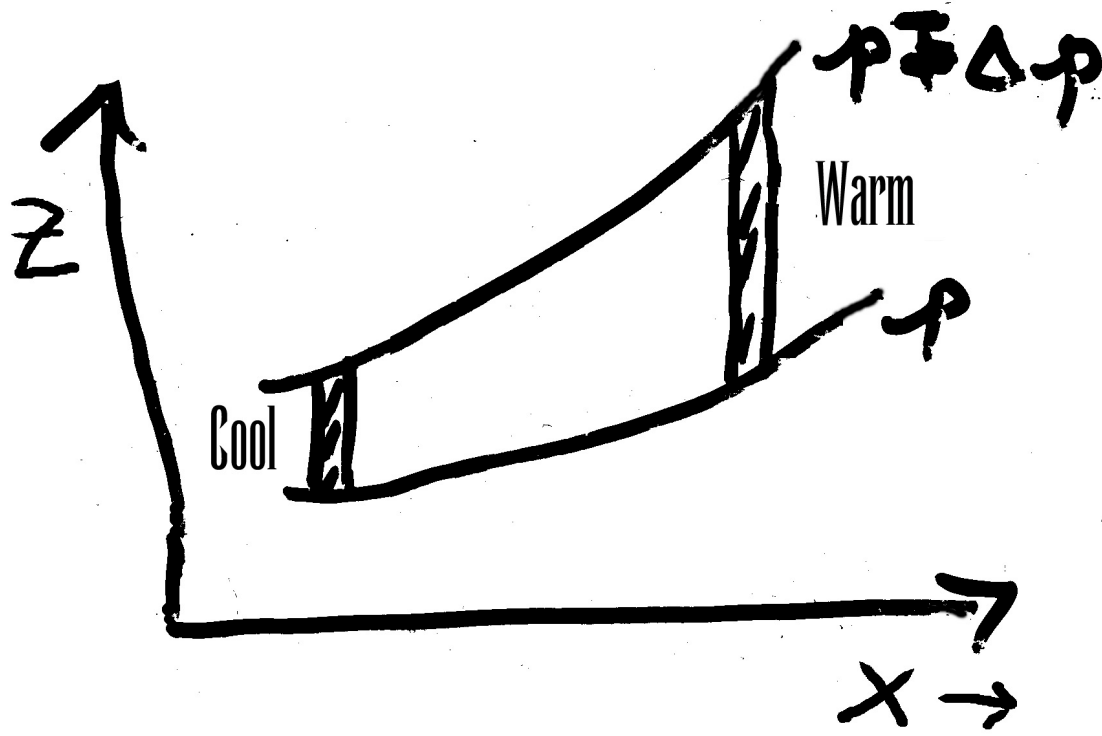


or

$$V_g = -\frac{1}{\rho f} \frac{\partial p}{\partial n} = -\frac{g_0}{f} \frac{\partial Z}{\partial n}$$

Tighter height grad-  
Stronger the winds.

## Thermal Wind



Geostrophic wind equations:

$$(1) \quad f v_g = -\frac{1}{\rho} \frac{\partial p}{\partial x}, \quad (2) \quad f u = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

Hydrostatic eq.      Eq. of state

$$(3) \quad g = -\frac{1}{\rho} \frac{\partial p}{\partial z}, \quad (4) \quad \rho = \frac{p}{R^* T}$$

Substitute (4) into (1), (2), (3)

$$(5) \quad f v_g = \frac{R^* T}{p} \frac{\partial p}{\partial x}; \quad (6) \quad f u_g = -\frac{R^* T}{p} \frac{\partial p}{\partial y}$$

$$(7) \quad g = -\frac{R^* T}{p} \frac{\partial p}{\partial z}$$

Cross differentiate between (5) and (7):

$$\begin{aligned} \frac{\partial}{\partial z} \left( \frac{f v_g}{T} \right) &= R^* \frac{\partial^2}{\partial z \partial x} (\ln p) \\ \frac{\partial}{\partial x} \left( \frac{g}{T} \right) &= -R^* \frac{\partial^2 \ln p}{\partial z \partial x} \end{aligned}$$

Adding above eqs. yields

$$(8) \quad \boxed{\frac{\partial}{\partial z} \left( \frac{f v_g}{T} \right) = -\frac{\partial}{\partial x} \left( \frac{g}{T} \right)}$$

Cross differentiate between (6) and (7)

$$\frac{\partial}{\partial z} \left( \frac{fu_g}{T} \right) = -R^* \frac{\partial^2(\ell np)}{\partial z \partial y}$$

$$\frac{\partial}{\partial y} \left( \frac{g}{T} \right) = -R^* \frac{\partial^2(\ell np)}{\partial z \partial y}$$

Subtracting 2<sup>nd</sup> from 1<sup>st</sup>:

$$(9) \quad \frac{\partial}{\partial z} \left( \frac{fu_g}{T} \right) = \frac{\partial}{\partial y} (g / T)$$

Completing differentiation of (8) and (9):

$$(10) \quad \frac{\partial v_g}{\partial z} = \frac{g}{fT} \frac{\partial T}{\partial x} \left[ + \frac{v}{T} \frac{\partial T}{\partial z} \right]$$

$$(11) \quad \frac{\partial u_g}{\partial z} = - \frac{g}{fT} \frac{\partial T}{\partial x} \left[ + \frac{u}{T} \frac{\partial T}{\partial z} \right]$$

Terms in  $\square$  are corrections for slope of isobaric surfaces and are small compared with 1<sup>st</sup> terms on RHS of (10) and (11).

Thus, thermal wind eqs.

$$\frac{\partial v_g}{\partial z} \square \frac{g}{fT} \frac{\partial T}{\partial x}; \quad \frac{\partial u_g}{\partial z} \square \frac{g}{fT} \frac{\partial T}{\partial y}$$

Vertical shear of horizontal wind is large where there are strong horizontal gradients in temperature (i.e., across polar front).

In Natural Coordinates

$$(V_g)_2 - Vg)_2 = \frac{\text{const. } \partial T}{f} \frac{\partial}{\partial n}$$

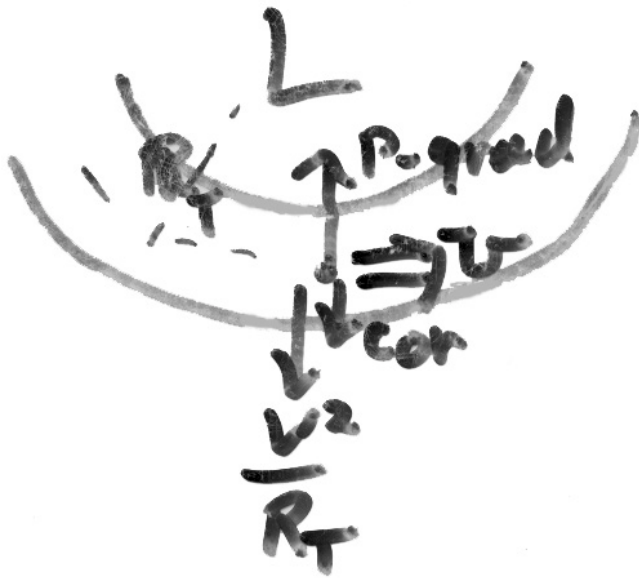
where T is the mean temperature of layer 1→2

Also,

$$(V_g)_2 - Vg)_2 = \frac{g_0}{f} \frac{\partial}{\partial n} (Z_2 - Z_1)$$

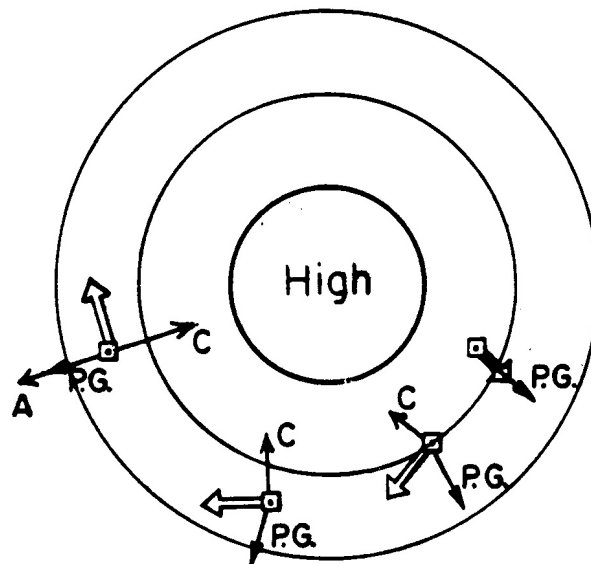
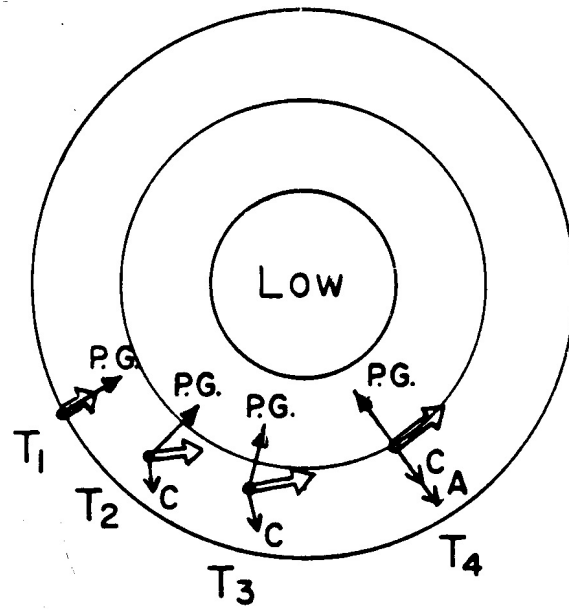
## Gradient Wind

Sharp troughs are often associated with the subgeostrophic flow. The flow can be balanced because of the large centripetal acceleration.



Apparent centrifugal force helps balance P-grad.

## Gradient Balance



Consider the equations of motion in cylindrical coordinates:

$$v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = f v_\theta - \frac{1}{\rho} \frac{\partial p}{\partial r}$$

$$v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} = -f v_r - \frac{1}{\rho r} \frac{\partial p}{\partial \theta}$$

Consider circular concentric isobars with centers at  $r = 0$ . Then  $\frac{\partial p}{\partial \theta} = 0$ ,

and for circular symmetry,  $\frac{\partial v_r}{\partial \theta} = \frac{\partial v_\theta}{\partial \theta} = 0$ , and  $V_r = 0$ . Then the first equation can be written

$$\frac{c^2}{R} + f c - \frac{1}{\rho} \frac{\partial p}{\partial r} = 0,$$

where  $c = v_\theta$  and  $R$  is the radius of the cyclone/anticyclone. Solutions to the above are

$$c = -\frac{fR}{2} \pm \sqrt{\frac{f^2 R^2}{4} + \frac{R}{\rho} \frac{\partial p}{\partial r}}$$

when  $\frac{\partial p}{\partial r}$  is positive (a low) the square root can never become imaginary so

that all values of pressure gradient are permitted. There is no theoretical restriction on the magnitude of the pressure gradient for a low. However,

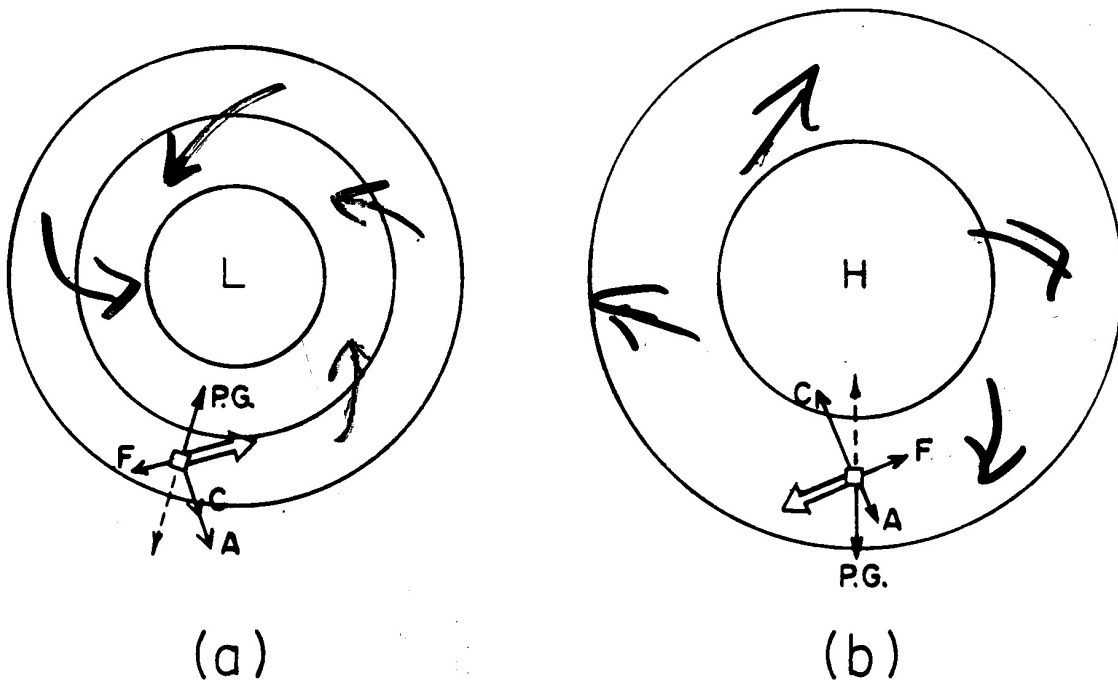
when  $\frac{\partial p}{\partial r} < 0$  (a high) the square root can become imaginary. For  $C$  to be

real,

$$\left| \frac{\partial p}{\partial r} \right| \leq \frac{\rho f^2 R}{4}$$

or a high may not exceed a value determined by the latitude and radius of curvature.

## Ekman Balance



The conditions for balance in the Ekman layer are that:

$$\frac{c^2}{R} + fc - \frac{1}{\rho} \frac{\partial p}{\partial r} + \underbrace{C_D c^2}_F = 0.$$

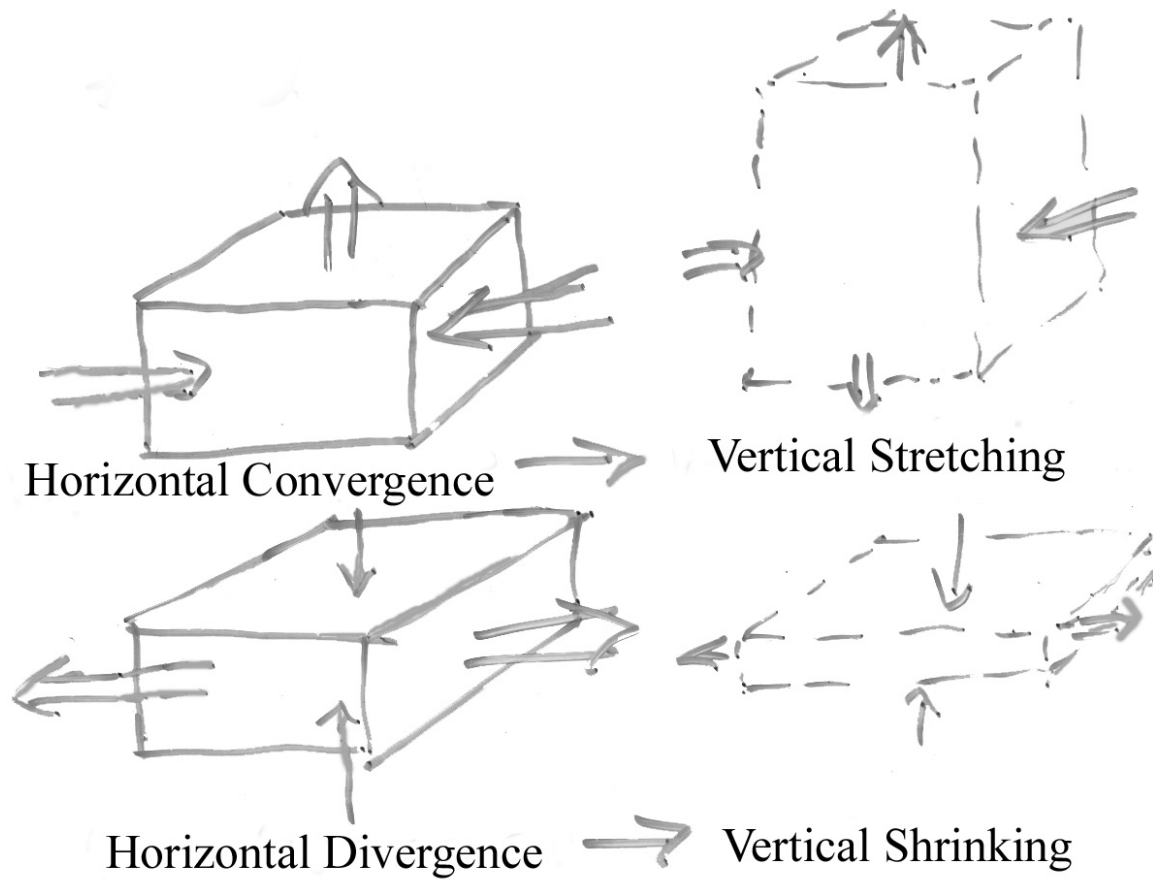
We see that friction decelerates the flow and turns the wind towards low pressure. This results in low-level divergence out of anticyclones and low-level convergence into cyclones.

In summary:

- Frictional acceleration acts directly opposite to the direction of the wind.
- Coriolis acceleration is perpendicular to the wind direction.
- Centripetal acceleration is also perpendicular to the instantaneous wind direction.

## Continuity Equation

The atmosphere behaves as in incompressible fluid



### Continuity Equation

$$\frac{\partial \rho}{\partial t} = - \left( \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right)$$

As an incompressible fluid:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$

or

$$\overbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}}^{\text{Hor. Div.}} = - \frac{\partial w}{\partial z}.$$

## Pressure Tendency Equation

The pressure at any height (z) is given by the weight of the air column above it:

$$-\int_p^0 dp = \int_0^p dp = p = g \int_z^\infty \rho dz.$$

The pressure tendency at z is

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial t} \left[ g \int_z^\infty \rho dz \right] = g \int_z^\infty \frac{\partial \rho}{\partial t} dz$$

But,

$$\frac{\partial \rho}{\partial t} = -\text{Div}_H (\rho \vec{V}_H) - \frac{\partial}{\partial z} (\rho w)$$

where  $\vec{V}_H = \sqrt{u^2 + v^2}$ .

or

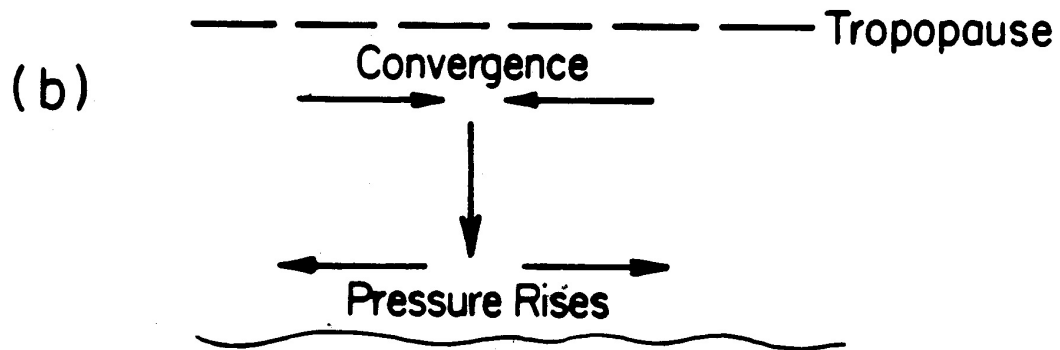
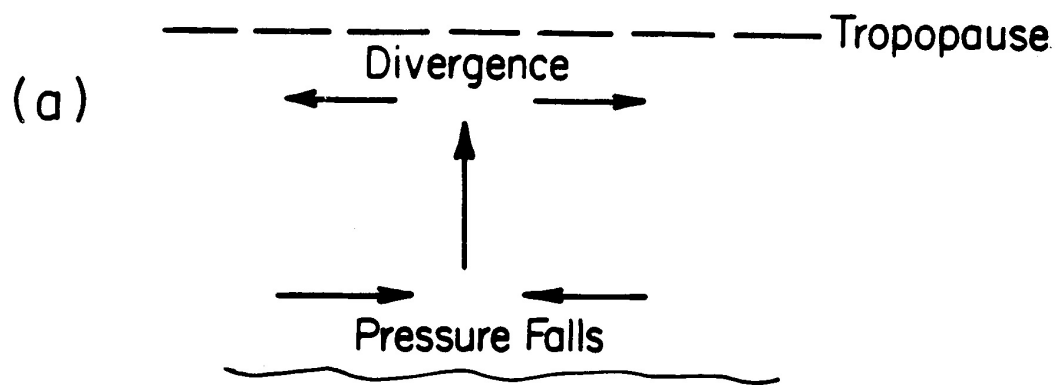
$$\left. \frac{\partial p}{\partial t} \right)_z = -g \int_z^\infty \operatorname{div}_H (\rho \vec{V}_H) - g \int_z^\infty \frac{\partial}{\partial z} (\rho w) dz$$

$$\left. \rho w \right)_\infty - \left. \rho w \right)_z$$

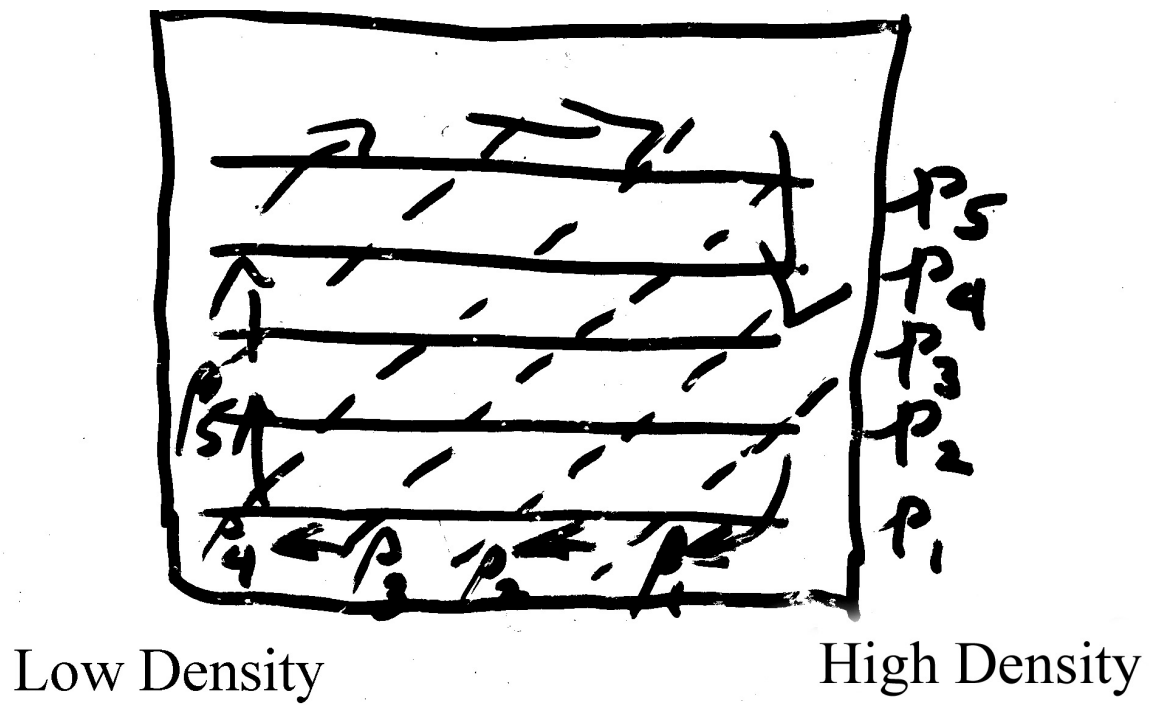
$$\boxed{\left. \frac{\partial p}{\partial t} \right)_z = -g \int_z^\infty \operatorname{div}_H (\rho \vec{V}_H) + g (\rho w)_z}$$

At the surface:  $(\rho w)_{z=0} \Rightarrow 0$ .

$$\left. \frac{\partial p}{\partial t} \right)_{z=0} = -g \int_{z=0}^\infty \operatorname{div}_H (\rho \vec{V}_H).$$



### Example of Baroclinic Atmosphere



K.E. is generated

Baroclinic systems:

- Cold fronts
- Sea breeze fronts
- Mtn. slope flows

| <b><i>Barotropic Atm.</i></b>       | <b><i>Baroclinic</i></b>             |
|-------------------------------------|--------------------------------------|
| 1) $\rho$ and p surfaces coincide   | 1) $\rho$ and p surfaces intersect   |
| 2) p and T surfaces coincide        | 2) p and T surfaces intersect        |
| 3) p and $\theta$ surfaces coincide | 3) p and $\theta$ surfaces intersect |
| 4) No geostrophic wind shear        | 4) Geostrophic wind shear            |
| 5) No large-scale w                 | 5) Large-scale w                     |

## Vorticity

Analogous to solid body angular momentum.

Vertical component:

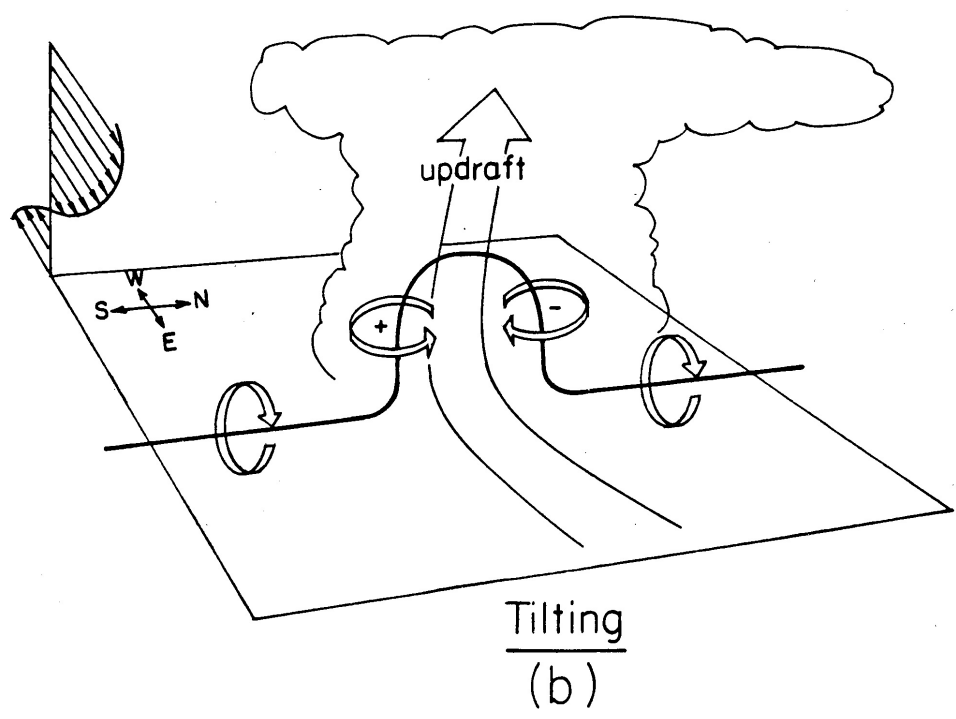
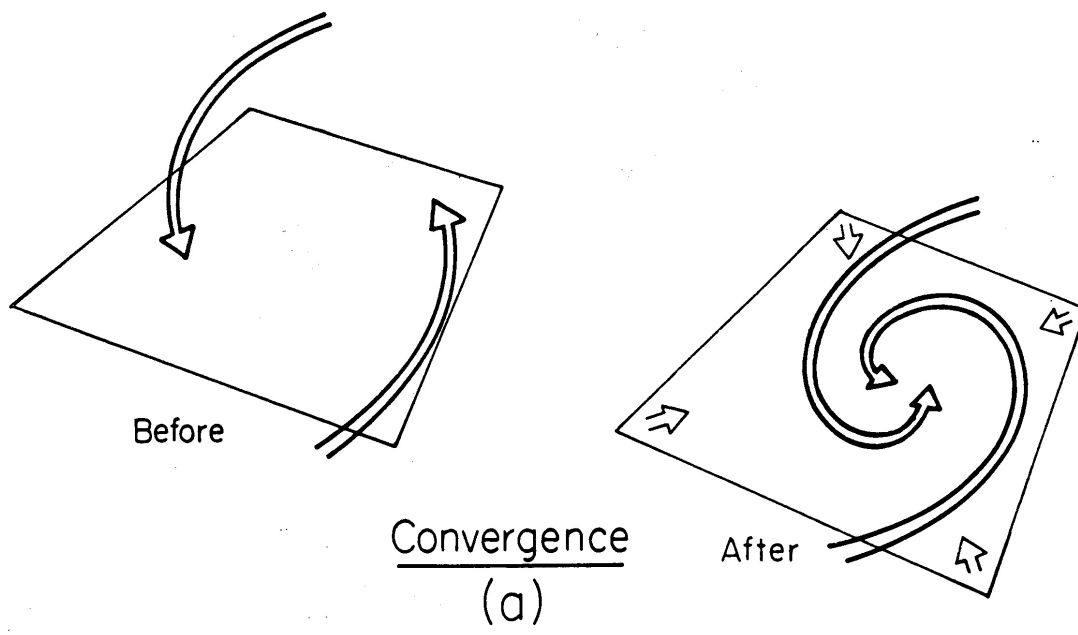
$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Consider equations of motion

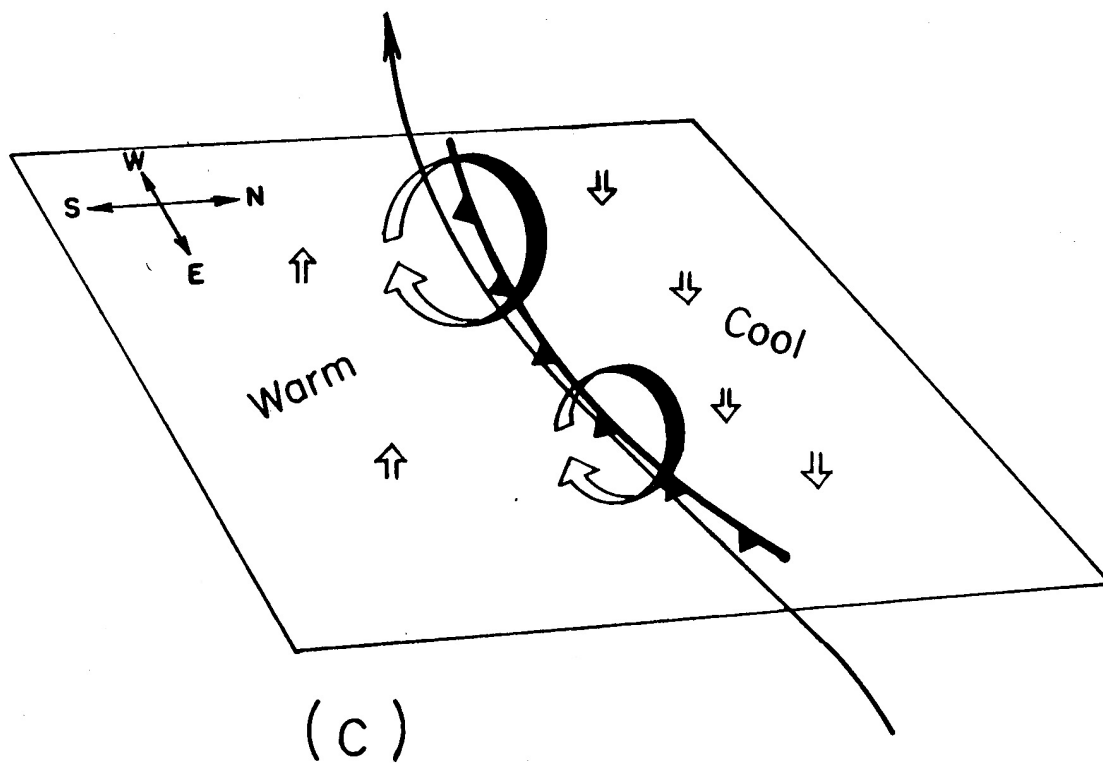
$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= -\alpha \frac{\partial p}{\partial x} + fv + F_x \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= -\alpha \frac{\partial p}{\partial y} - fu + F_y \end{aligned}$$

Take vector cross product or take partial derivative with respect to x of 2<sup>nd</sup> equation, and subtract partial derivative with respect to y of the 1<sup>st</sup> equation and rearrange:

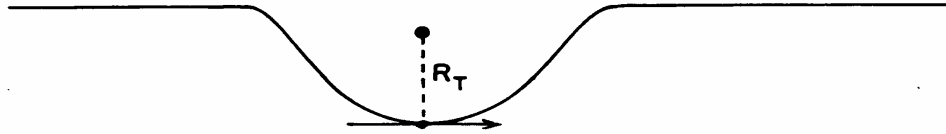
$$\begin{aligned} & \frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + v \frac{2f}{2y} \\ & \quad \text{(divergence)} \quad \quad \quad \text{(tilting)} \\ &= -(f + \zeta) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \left[ \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} \right] \\ & \quad \text{solenoidal or baroclinic} \quad \quad \text{friction} \\ &= - \left[ \frac{\partial \alpha}{\partial x} \frac{\partial p}{\partial y} - \frac{\partial \alpha}{\partial y} \frac{\partial p}{\partial x} \right] + \left[ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right] \end{aligned}$$



## Baroclinicity



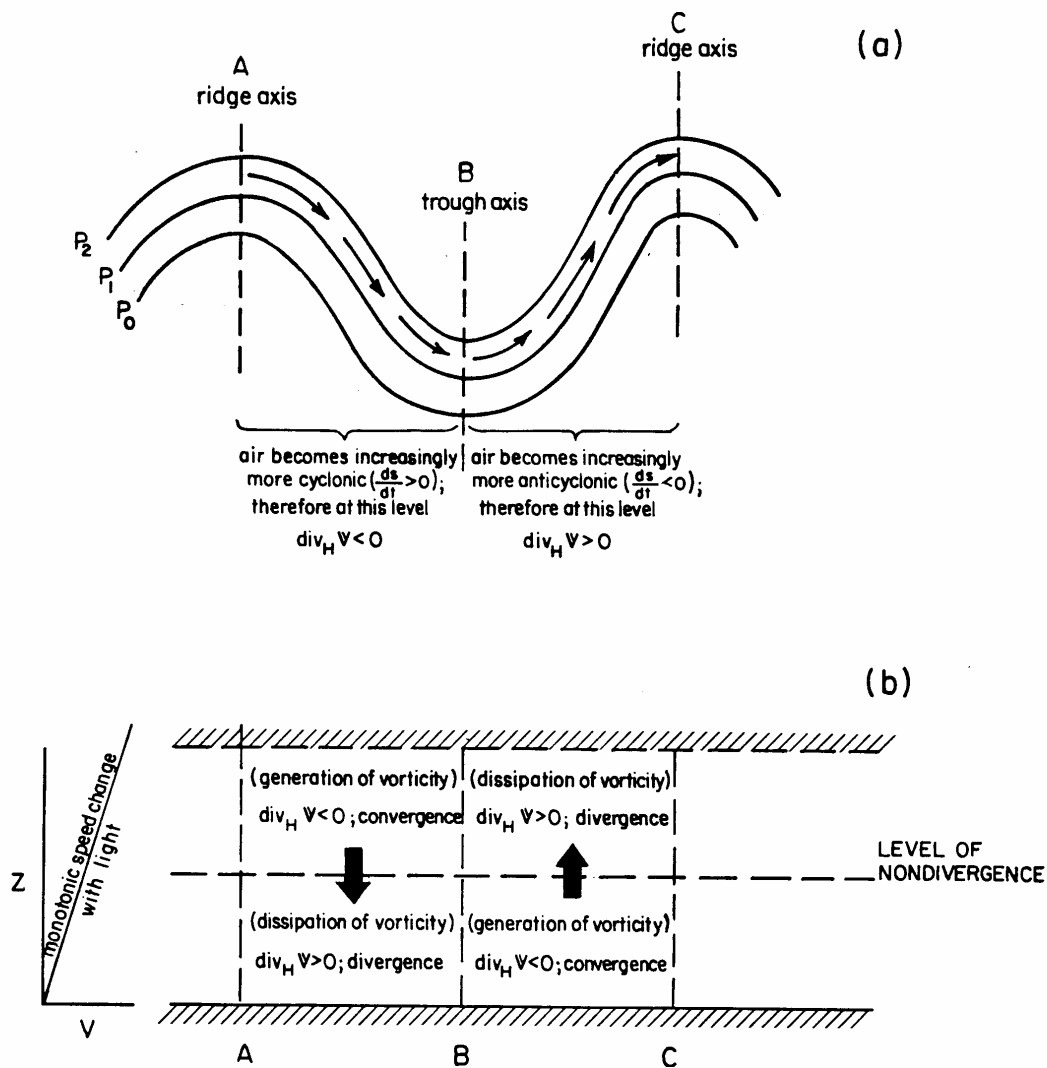
Rotational Vorticity  $\frac{V}{R_T}$



Shear Vorticity  $-\frac{\partial V}{\partial n}$



Examples of rotational and shear cyclonic vorticity illustrated in natural coordinates.



Schematic illustration of the inferred change of vorticity and resultant motion (b) as an air parcel in gradient wind balance moves through a constant pressure gradient wind field in the upper troposphere given in (a).

## The Omega Equation

We desire to form a single equation that combines the vorticity equation and the first law of thermodynamics to describe the vertical motion pattern above the surface associated with extratropical cyclones.

We follow procedures used to derive the vorticity equation previously except that we do so on a constant pressure surface. For compactness, we write in vector notation:

$$\frac{\partial \zeta_p}{\partial t} + \bar{\mathbf{V}} \cdot \nabla_p (\zeta_p + f) \frac{\partial \omega}{\partial p},$$

where  $\zeta_p$  is the relative vorticity on a constant pressure surface. We have

also neglected tilting and fraction terms. The quantity  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$  on a

constant pressure surface is replaced by  $\frac{\partial \omega}{\partial p}$ , where  $\omega = dp/dt = -\rho g w$ .

Introducing the concept of geostrophic vorticity,  $\zeta_g$ , and giving the vector form of geostrophic velocity as

$$\vec{V}_g = \frac{g}{f} \vec{k} \times \nabla_p Z,$$

then

$$\zeta_g = \vec{\nabla} \times \vec{V}_g = \frac{g}{f} \nabla_p^2 Z.$$

Substituting  $\zeta_g$  into the vorticity tendency equation and letting  $\zeta p = \zeta g$ ,

$$\frac{\partial}{\partial t} \left( \frac{g}{f} \nabla_p^2 Z \right) + \vec{V} \cdot \nabla p (\zeta_g + f) = (\zeta_g + f) \frac{\partial \omega}{\partial p}.$$

Writing the first law of thermodynamics as:

$$C_p \frac{d \ln \theta}{dt} = Q/T,$$

where Q represents sensible heating.

$$C_p \left[ \frac{\partial \ln \theta}{\partial t} + u \frac{\partial \ln \theta}{\partial x} + v \frac{\partial \ln \theta}{\partial y} + \omega \frac{\partial \ln \theta}{\partial p} \right] = Q/T$$

Since using the gas law:

$$\theta = T [1000/p]^{R_d/C_p} = \frac{p\alpha}{R_d} \left( \frac{1000}{p} \right)^{R_d/C_p}$$

then

$$\frac{\partial \ln \theta}{\partial x} = \frac{\partial \ln \alpha}{\partial x}; \frac{\partial \ln \theta}{\partial y} = \frac{\partial \ln \alpha}{\partial y}; \frac{\partial \ln \theta}{\partial t} = \frac{\partial \ln \alpha}{\partial t}$$

on a constant pressure surface and

$$C_p \left[ \frac{\partial \alpha}{\partial t} + u \frac{\partial \alpha}{\partial x} + v \frac{\partial \alpha}{\partial y} + \omega \alpha \frac{\partial \ln \theta}{\partial p} \right] = \frac{\alpha}{T} Q.$$

From the hydrostatic relation in a pressure coordinate framework (i.e.,  $\frac{\partial z}{\partial p} = -\alpha/g$ ):

$$\alpha = -g \frac{\partial z}{\partial p}$$

so that the above can also be written as:

$$C_p \left[ -\frac{\partial}{\partial t} \left( g \frac{\partial z}{\partial p} \right) - u \frac{\partial}{\partial x} \left( g \frac{\partial z}{\partial p} \right) - v \frac{\partial}{\partial y} \left( g \frac{\partial z}{\partial p} \right) + \omega \alpha \frac{\partial \ln \theta}{\partial p} \right] = \frac{\alpha}{T} Q$$

By convention:

$$\sigma = -\alpha \frac{\partial \ln \theta}{\partial p} = g \frac{\partial z}{\partial p} \frac{\partial \ln \theta}{\partial p}$$

is defined so that the above becomes, after rearranging:

$$\frac{\partial}{\partial t} \left( -g \frac{\partial z}{\partial p} \right) - \vec{V} \cdot \vec{\nabla}_p \left( g \frac{\partial z}{\partial p} \right) - \omega \sigma = \frac{\alpha}{C_p T} Q = \frac{R_d}{p C_p} Q.$$

Performing the operation  $\partial/\partial p$ :

$$g\nabla_p^2 \frac{\partial}{\partial t} \frac{\partial z}{\partial p} + f \frac{\partial}{\partial p} [\vec{V} \cdot \nabla_p (\xi_g + f)] = f (f + \xi_g) \frac{\partial^2 \omega}{\partial p^2};$$

performing the operation  $\nabla_p^2$  and assuming that  $\sigma$  is a function of pressure only yields:

$$-g\nabla_p^2 \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial p} \right) - \nabla_p^2 \left[ \vec{V} \cdot \nabla_p \left( g \frac{\partial z}{\partial p} \right) \right] - \sigma \nabla_p^2 \omega = \frac{R_d}{p C_p} \nabla_p^2 Q.$$

Adding the last two equations produces:

$$f \frac{\partial}{\partial p} [\vec{V} \cdot \nabla_p (\xi + f)] - \nabla_p^2 \left[ \vec{V} \cdot \nabla_p \left( g \frac{\partial z}{\partial p} \right) \right] - \sigma \nabla_p^2 \omega = \frac{R_d}{C_p p} \nabla_p^2 Q + f (f + \xi_g) \frac{\partial^2 \omega}{\partial p^2}.$$

Since  $\partial z/\partial p = -\alpha/g = -RT/gp$ , this relation can also be written as:

$$\sigma \nabla_p^2 \omega + f (f + \xi_g) \frac{\partial^2 \omega}{\partial p^2} = f \frac{\partial}{\partial p} [\vec{V} \cdot \vec{\nabla}_p (\xi_g + f)] + \frac{R_d}{p} \nabla_p^2 [\vec{V} \cdot \vec{\nabla}_p T] - \frac{R_d}{C_p p} \nabla_p^2 Q.$$

This equation is called the *Omega equation* and represents a diagnostic second order differential equation for  $\frac{d\omega}{dt}$ .

The three terms on the right side represent the following:

$$\frac{\partial}{\partial p} [\vec{V} \cdot \vec{\nabla}_p (\xi_g + f)] \longrightarrow \text{vertical variation of the advection of absolute vorticity on a constant pressure surface.}$$

$$\nabla_p^2 [\vec{V} \cdot \vec{\nabla}_p T] \longrightarrow \text{the curvature of the advection of temperature on a constant pressure surface.}$$

$$\nabla_p^2 Q \longrightarrow \text{the curvature of diabatic heating on a constant pressure surface.}$$

These three terms can be interpreted more easily.

Using the relation between  $\partial/\partial p$  and  $\partial/\partial z$ , and our observation that  $\nabla^2 \omega \sim w$ ,

$$w \sim \frac{\partial}{\partial p} [\vec{V} \cdot \vec{\nabla}_p (\xi_g + f)] \sim -\frac{\partial}{\partial z} [\vec{V} \cdot \vec{\nabla}_p (\xi_g + f)]$$

In most situations in the atmosphere, the vorticity advection is much smaller in the lower troposphere than in the middle and upper troposphere since  $\vec{V}$  and  $\xi_g$  are usually smaller near the surface. We have shown that on the synoptic scale, cold air towards the poles requires that  $\vec{V}$  becomes more positive with height.

Using this observation of the behavior of  $\vec{V}$  and  $\xi_g$  with height:

$$w \sim -\vec{V} \cdot \vec{\nabla}_p (\xi_g + f)$$

In other words, vertical velocity is proportional to vorticity advection. Since upper-level vorticity patterns are usually geographically the same as at midtropospheric levels (since troughs and ridges are nearly vertical in the upper troposphere, the 500 mb level is generally chosen to estimate vorticity advection. This level is also close to the level of nondivergence in which creation or dissipation of relative vorticity is small, so that the conservation of absolute vorticity is a good approximation.

Thus for the Northern Hemisphere where  $\xi_g > 0$  for cyclonic vorticity,

|                                                                                               |
|-----------------------------------------------------------------------------------------------|
| $w > 0$ if $-\vec{V} \cdot \vec{\nabla}_p (\xi_g + f) > 0$ positive vorticity advection (PVA) |
| $w < 0$ if $-\vec{V} \cdot \vec{\nabla}_p (\xi_g + f) < 0$ negative vorticity advection (NVA) |

To generalize this concept to the southern hemisphere, PVA should be called cyclonic vorticity advection; NVA should be referred to as anticyclonic vorticity. The curvature of the advection of temperature on a constant pressure term can be represented as:

$$\nabla_p^2 [\vec{V} \cdot \vec{\nabla}_p T] \sim -k^2 B \sin kx$$

where  $B$  is a constant. Therefore,

$$\vec{V} \cdot \vec{\nabla}_p T \sim B \sin kx$$

Since:

$$w \sim \nabla_p^2 [\vec{V} \cdot \vec{\nabla}_p T]$$

then

$$w \sim -\vec{V} \cdot \vec{\nabla}_p T.$$

Thus,

|                                                                 |
|-----------------------------------------------------------------|
| $w > 0$ if $-\vec{V} \cdot \vec{\nabla}_p T > 0$ warm advection |
| $w < 0$ if $-\vec{V} \cdot \vec{\nabla}_p T < 0$ cold advection |

The 700 mb surface is often used to evaluate the temperature advection patterns since the gradients of temperature are often larger at this height than higher up and the winds are significant in speed. The 850 mb height can be used (when the terrain is low enough) although the values of  $\vec{V}$  are often substantially smaller.

Finally, since  $\nabla_p^2 Q \sim -k^2 C \sin kx$  can be assumed in this form,  $w \sim -\nabla_p^2 Q$ , and  $Q \sim w$  results.

Therefore,

|                          |
|--------------------------|
| $w > 0$ diabatic heating |
| $w < 0$ diabatic cooling |

An example of diabatic heating on the synoptic scale is deep cumulonimbus activity. An example of diabatic cooling is longwave radiative flux divergence.

In summary, the preceding analysis suggests the following relation between vertical motion, vorticity and temperature advection, and diabatic heating.

$$w > 0 \left\{ \begin{array}{l} \text{positive vorticity advection} \\ \text{warm advection} \\ \text{diabatic heating.} \end{array} \right. \quad w < 0 \left\{ \begin{array}{l} \text{negative vorticity advection} \\ \text{cold advection} \\ \text{diabatic cooling.} \end{array} \right.$$

When combinations of terms exist which would separately result in different signs of the vertical motion (e.g., positive vorticity advection with cold advection), the resultant vertical motion will depend on the relative magnitudes of the individual contributions. Also, remember that this relation for vertical motion is only accurate as long as the assumptions used to derive the Omega equation are valid.

Using synoptic analyses the following rules of thumb usually apply:

- i) vorticity advection: evaluate at 500 mb.
- ii) temperature advection: evaluate at 700 mb; at elevations near sea level, also evaluate at 850 mb.
- iii) diabatic heating: contribution of major importance in synoptic weather patterns (Particularly cyclogenesis) are areas of deep cumulonimbus. Refer to geostationary satellite imagery and radar for determination of locations of deep convection.

### Petterssen's development equation

The vorticity equation can be written as:

$$\frac{\partial(\xi_z + f)}{\partial t} + \vec{V}_H \cdot \nabla p(\xi_z + f) = 0$$

if vertical advection of absolute vorticity, the tilting term and the solenoidal term are ignored. We assumed that the above equation is valid at the level of nondivergence ( $\sim 500$  mb).  $\vec{V}_H$  is the wind on the pressure surface. Since, if the wind is in geostrophic balance:

$$\vec{V}_{H500} = \vec{V}_{H_{SFC}} + \Delta\vec{V}_g$$

where  $\Delta\vec{V}_g$  is the geostrophic wind shear. Thus,

$$(\xi_z + f)_{500} = (\xi_z + f)_{SFC} + (\xi_z + f)_T$$

since  $\nabla \times \vec{V}_{H500} = (\nabla \times \vec{V}_{H_{SFC}}) + (\nabla \times \Delta\vec{V})$ . We can write the vorticity equation as:

$$\frac{\partial(\xi_z + f)_{SFC}}{\partial t} = -\vec{V}_{H500} \cdot \nabla_p(\xi_z + f)_{500} - \frac{\partial(\xi_z + f)_T}{\partial t}$$

From the thermal wind equation,

$$\nabla \times \Delta\vec{V}_g = \frac{g}{f} \nabla_p^2 (\Delta z)$$

where  $\Delta z = z_{500} - z_G$  with  $z_{500}$  the 500 mb height and  $z_G$  the surface elevation so that,

$$\frac{\partial(\xi_z + f)_T}{\partial t} = \frac{g}{f} \nabla_p^2 \frac{\partial(\Delta z)}{\partial t}$$

Integrating between the surface pressure,  $p_{SFC}$ , and 500 mb yields, after rearranging:

$$-g \int_{p_{SFC}}^{500} \frac{\partial}{\partial t} \left( \frac{\partial z}{\partial p} \right) dp = -g \frac{\partial}{\partial t} \int_{z_G}^{z_{500}} dz = -g \frac{\partial(\Delta z)}{\partial t} = \int_{p_{SFC}}^{500 \text{mb}} \left( \vec{V} \cdot \nabla_p \left( g \frac{\partial z}{\partial p} \right) + \omega \sigma + \frac{R}{p C_p} Q \right) dp$$

Performing  $\nabla_p^2$  on the above equation, substituting into the vorticity equation yields:

$$\begin{aligned} \frac{\partial(\xi_z + f)_{SFC}}{\partial t} = & -\vec{V}_{H500} \cdot \nabla_p(\xi_z + f)_{500} + \frac{g}{f} \nabla_p^2 \int_{p_{SFC}}^{500} \vec{V}_H \cdot \nabla_p \left( \frac{\partial z}{\partial p} \right) dp \\ & + \frac{\nabla_p^2}{f} \int_{p_{SFC}}^{500} \omega \sigma dp + \frac{R \nabla_p^2}{f C_p} \int_{p_{SFC}}^{500} \frac{Q}{p} dp \end{aligned}$$

This is the Petterssen development equation for the change of surface absolute vorticity due to:

- $-\vec{V}_{H_{500}} \cdot \nabla_p (\xi_z + f)_{500}$  : horizontal vorticity advection at 500 mb.
- $\frac{g}{f} \nabla_p^2 \cdot \int_{p_{SFC}}^{500} \vec{V}_H \cdot \nabla_p \left( \frac{\partial z}{\partial p} \right) dp = -\frac{R}{f} \nabla_p^2 \int_{p_{SFC}}^{500} \frac{\vec{V}_H \cdot \nabla_p}{p} (T) dp$  : proportional to a pressure-weighted horizontal temperature advection between the surface and 500 mb.
- $\frac{\nabla_p^2}{f} \int_{p_{SFC}}^{500mb} \sigma \omega dp$  : proportional to vertical motion through the layer.
- $\frac{R \nabla_p^2}{f C_p} \int \frac{Q}{p} dp$  : proportional to a pressure-weighted diabatic heating pattern.

## The Q Vector

In order to keep the mathematical development as simple as possible we will consider the Q-Vector formulation of the omega equation only for the case in which  $\beta$  is neglected. This is usually referred to as an *f plane* because it is equivalent to approximating the geometry by a Cartesian planar geometry with constant rotation.

On the *f plane* the quasi-geostrophic prediction equations may be expressed simply as follow:

$$\frac{D_g u_g}{Dt} - f_0 v_a = 0 \quad (Q1)$$

$$\frac{D_g v_g}{Dt} + f_0 u_a = 0 \quad (Q2)$$

$$\frac{D_g T}{Dt} - S_\rho \omega = 0 \quad (Q3)$$

These are coupled by the thermal wind relationship

$$p \frac{\partial u_g}{\partial p} = \frac{R}{f_0} \frac{\partial T}{\partial y}, \quad p \frac{\partial v_g}{\partial p} = -\frac{R}{f_0} \frac{\partial T}{\partial x} \quad (Q4)$$

We now eliminate the time derivatives by first taking

$$p \frac{\partial}{\partial p} (Q1) - \frac{R}{f_0} \frac{\partial}{\partial y} (Q3)$$

to obtain

$$p \frac{\partial}{\partial p} \left[ \frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} - f_0 v_a \right] - \frac{R}{f_0} \frac{\partial}{\partial y} \left[ \frac{\partial T}{\partial t} + u_g \frac{\partial T}{\partial x} + v_g \frac{\partial T}{\partial y} - S_p \omega \right] = 0$$

Using the chain rule of differential equations, this may be rewritten as

$$\begin{aligned} \frac{RS_p}{f_0} \frac{\partial \omega}{\partial y} - f_0 p \frac{\partial v_a}{\partial p} = & - \left( \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \right) \left( p \frac{\partial u_g}{\partial p} - \frac{R}{f_0} \frac{\partial T}{\partial y} \right) \\ & - p \left[ \frac{\partial u_g}{\partial p} \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p} \frac{\partial u_g}{\partial y} \right] + \frac{R}{f_0} \left[ \frac{\partial u_g}{\partial y} \frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial T}{\partial y} \right] \end{aligned}$$

But, by the thermal wind relation (Q4) the term in parenthesis on the right-hand side vanishes and

$$-p \left[ \frac{\partial u_g}{\partial p} \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p} \frac{\partial u_g}{\partial y} \right] = -\frac{R}{f_0} \left[ \frac{\partial T}{\partial y} \frac{\partial u_g}{\partial x} - \frac{\partial T}{\partial x} \frac{\partial u_g}{\partial y} \right]$$

Using these facts, plus the fact that

$$\partial u_g / \partial x + \partial v_g / \partial y = 0$$

we finally obtain the simplified form

$$\sigma \frac{\partial \omega}{\partial y} - f_0^2 \frac{\partial v_a}{\partial p} = -2Q_2 \quad (Q5)$$

where

$$Q_2 \equiv -\frac{R}{p} \left[ \frac{\partial u_g}{\partial y} \frac{\partial T}{\partial y} + \frac{\partial v_g}{\partial y} \frac{\partial T}{\partial y} \right] = -\frac{R}{p} \frac{\partial V_g}{\partial y} \cdot \nabla T$$

Similarly, if we take

$$p \frac{\partial}{\partial p} (Q2) + \frac{R}{f_0} \frac{\partial}{\partial x} (Q3)$$

followed by application of (Q4) we obtain

$$\sigma \frac{\partial \omega}{\partial x} - f_0^2 \frac{\partial u_a}{\partial p} = -2Q_1 \quad (\text{Q6})$$

where

$$Q_1 \equiv -\frac{R}{p} \left[ \frac{\partial u_g}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial T}{\partial y} \right] = -\frac{R}{p} \frac{\partial V_g}{\partial x} \cdot \nabla T$$

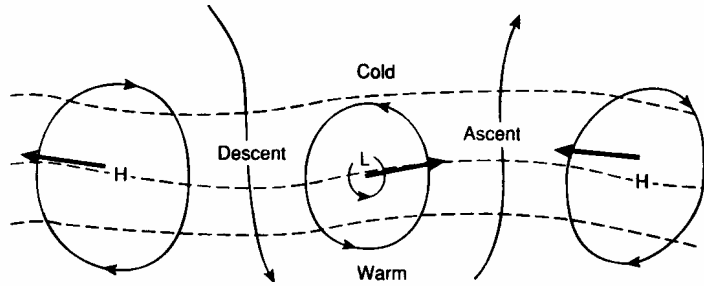
If we now take  $\partial (\text{Q6}) / \partial x + \partial (\text{Q5}) / \partial y$  and use the continuity equation to eliminate the ageostrophic wind, we obtain the Q-vector form of the omega equation:

$$\sigma \nabla^2 \omega + f_0^2 \frac{\partial^2 \omega}{\partial p^2} = -2 \nabla \cdot Q$$

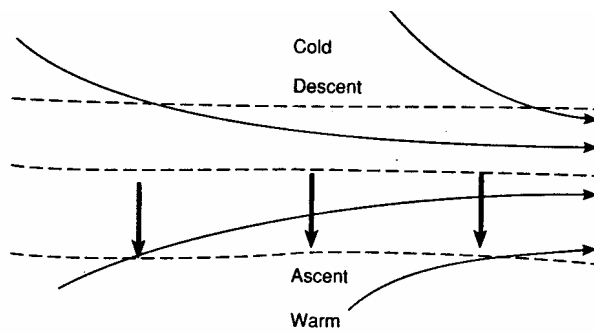
where

$$Q \equiv (Q_1, Q_2) = \left( -\frac{R}{p} \frac{\partial V_g}{\partial x} \cdot \nabla T, -\frac{R}{p} \frac{\partial V_g}{\partial y} \cdot \nabla T \right)$$

This shows that on the  $f$  plane vertical motion is forced only by the divergence of  $Q$ . Unlike the traditional form of the omega equation, the  $Q$ -vector form does not have forcing terms that partly cancel. The forcing of  $\omega$  can be represented simply by the pattern of the  $Q$ -vector. Hence, regions where  $Q$  is convergent (divergent) correspond to ascent (descent).



Q vectors (bold arrow) for idealized pattern of isobars (solid) and isotherms (dashed) for a family of cyclones and anticyclones. (After Sanders and Hoskins, 1990).



Orientation of Q vectors (bold arrows) for confluent (jet entrance flow). Dashed lines are isotherms. (After Sanders and Hoskins, 1990).

## Potential Vorticity

The earliest mention of potential vorticity is found in a paper by Rossby (1940), in which he was exploring the character of flow patterns in the atmosphere and how they change. Rossby defines potential vorticity in the shallow-water framework as

$$PV = \frac{\zeta + f}{h}, \quad \text{PVI}$$

where  $\zeta$  is the relative vorticity,  $f$  the Coriolis parameter, and  $h$  the depth of the fluid. For adiabatic, frictionless flow this quantity is conserved. Thus if we stretch the fluid (i.e., increase  $h$ ) we must increase the absolute vorticity. For zonal flows this means we must increase  $\zeta$ , resulting in increased cyclonic flow. This compact quantity combines information about mass and flow fields in one variable and has been used to explain, among other things, lee cyclogenesis, as a column of air is stretched after crossing a mountain barrier.

### *Ertel's potential vorticity*

A more general definition of potential vorticity is found in Ertel (1942). Starting with a three-dimensional vector form of the equations of motion for frictionless flow, along with the thermodynamic equation for adiabatic motion and the mass continuity equation, Ertel derived the conservation principle,

$$\frac{d}{dt}(\alpha \zeta_a \cdot \nabla \psi) = 0 \quad PV2$$

where  $\alpha$  is the specific volume (i.e.,  $1/\rho$ ),  $\zeta_a$  is the absolute vorticity vector, and  $\psi$  is any conservative thermodynamic variable. In meteorology we generally use potential temperature,  $\theta$ , for the conserved thermodynamic quantity. An example of an alternate conserved scalar is found in Lait (1994).

Using  $\theta$ , our equation becomes

$$\frac{d\mathcal{P}}{dt} = \frac{d}{dt}(\alpha \zeta_a \cdot \nabla \theta) = 0. \quad PV3$$

The conserved quantity  $\mathcal{P} = \alpha \zeta_a \cdot \nabla \theta$  is referred to as Ertel's potential vorticity.<sup>1</sup> It is important to note that the only assumptions are for frictionless, adiabatic flow; an extension of Ertel's theory to include diabatic and frictional effects is straightforward.

---

<sup>1</sup>We will use  $PV$  for general use in describing potential vorticity-like quantities, i.e., a vorticity divided by some measure of fluid depth or mass. The symbol  $\mathcal{P}$  will be used only for Ertel's  $PV$  as defined above.  $EPV$  is sometimes used in the literature for  $\mathcal{P}$ .

Starting with the vector equation of motion which includes friction, and a thermodynamic equation with a term representing diabatic processes, we obtain

$$\frac{d\mathcal{P}}{dt} = \mathcal{H} + \mathcal{F} \quad \text{PV4}$$

where  $\mathcal{H} = (1/\rho)\zeta_a \cdot \nabla Q$ , and  $Q = d\theta/dt$  is a diabatic potential temperature source (material change in potential temperature) and  $\mathcal{F} = (1/\rho)\nabla\theta \cdot \nabla \times \mathbf{F}$  is the material change in  $\mathcal{P}$  due to frictional effects.

Expanding the dot product in Eq. (23), we find that  $\mathcal{P}$  is a scalar quantity made up of three parts, i.e.,

$$\mathcal{P} = \frac{1}{\rho} \left[ \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \frac{\partial \theta}{\partial x} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \frac{\partial \theta}{\partial y} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f \right) \frac{\partial \theta}{\partial z} \right]. \quad \text{PV5}$$

The first two terms represent the product of horizontal vorticity and horizontal gradients of  $\theta$ , (baroclinic terms), while the final term is the product of the vertical component of absolute vorticity and static stability (vertical  $\theta$  gradients).

The units associated with  $\mathcal{P}$  merit comment; from <sup>PV5</sup>(25) these are a rather unintuitive  $\text{m}^2 \text{s}^{-1} \text{K kg}^{-1}$ . Pedlosky (1979, pg. 41) states: "the name seems poorly chosen, for  $PV$  does not even have the dimensions of vorticity". Bleck (1973) says of Ertel's  $PV$ : "we are as comfortable as Rossby about the physical meaning of this dimensionally awkward quantity". The most common units used for  $\mathcal{P}$  today are those suggested by Hoskins et al. (1985). They define  $1 \text{ PVU} = 10^{-6} \text{ m}^2 \text{s}^{-1} \text{K kg}^{-1}$ . Using this definition, ambient tropospheric  $\mathcal{P}$  values are typically less than 0.5 PVU, with a typical tropopause value of 2 PVU. It can be argued that redefining  $\mathcal{P}$ , such that only units of  $\text{s}^{-1}$  (i.e., the units of vorticity) remain, is more consistent with the appellation "potential" vorticity. In this work, we will abide by recent tradition and use PVU.

While <sup>PV4</sup>(24) is expressed in material form, it can also be written in flux form. Decomposing the total time derivative on the left-hand side of <sup>PV4</sup>(24), multiplying through by  $\rho$  and using mass continuity yields

$$\frac{\partial(\rho\mathcal{P})}{\partial t} = -\nabla \cdot [\rho\mathbf{u}\mathcal{P} - \zeta_a Q - K\theta] \quad \text{PV6}$$

where  $\mathbf{u}$  is the three-dimensional velocity vector and  $\mathbf{K} = \nabla \times \mathbf{F}$  is the frictional force-curl. The predicted quantity  $\rho\mathcal{P}$  has been referred to as "potential vorticity substance".

The first term in brackets on the right-hand side of <sup>PVG</sup>(2.6) is the advective flux of  $\rho\mathcal{P}$ , while the second and third terms in brackets are the non-advective flux.

---

In the four decades following Ertel's original derivation, the usefulness of  $PV$  was not widely explored, however several references can be found in the literature. In a series of papers, Kleinschmidt (1950a,b, 1951) and later Eliassen and Kleinschmidt (1957) applied Ertel's theorem to cyclone theory, showing that surface cyclones were accompanied by a positive  $PV$  anomaly (i.e., high- $PV$  air relative to the environment) aloft. Kleinschmidt was also apparently the first to recognize that  $PV$  has an important further use. Under certain constraints, information about the mass and wind fields can <sup>be</sup> diagnosed from  $PV$ .

Bleck (1973) expanded

on Kleinschmidt's work with a *PV*-predicting model using isentropic vertical coordinates. The model successfully predicts surface cyclogenesis with the approach of high-*PV* air aloft, in agreement with the analysis of Kleinschmidt.

Other early work involves the use of *PV* as a tracer of stratospheric air. Since *PV* is a function of static stability, regions with strong static stability should also be regions of high potential vorticity (assuming the absolute vorticity doesn't vary greatly from  $f$ ). The stratosphere has been recognized as a possible source region or reservoir of high-*PV* air; above the tropopause  $\partial\theta/\partial z$  rapidly increases and hence so does *PV*. An early example of such work can be found in Staley (1960) who explored the possibility that *PV* could be used as a tracer to forecast the transfer of radioactive debris. Other studies investigating *PV* and other stratospheric tracers near tropopause folds using aircraft data have been carried out by Shapiro (1976) and Danielson et al. (1987).

Much of the early work on *PV* and the invertibility principle mentioned above is summarized in the review by Hoskins et al. (1985). The authors demonstrate the usefulness of *PV* when viewed on isentropic maps for both cutoff cyclones and blocking highs, and point to conceptual links between the two phenomena. An important concept they discuss is the "action at a distance" idea associated with the invertibility principle.

Circulations and mass perturbations associated with a given *PV* anomaly extend both horizontally and vertically well beyond the anomaly itself. Note that the positive *PV* anomaly is made up of both anomalously high absolute vorticity and static stability (Fig. 6V1).



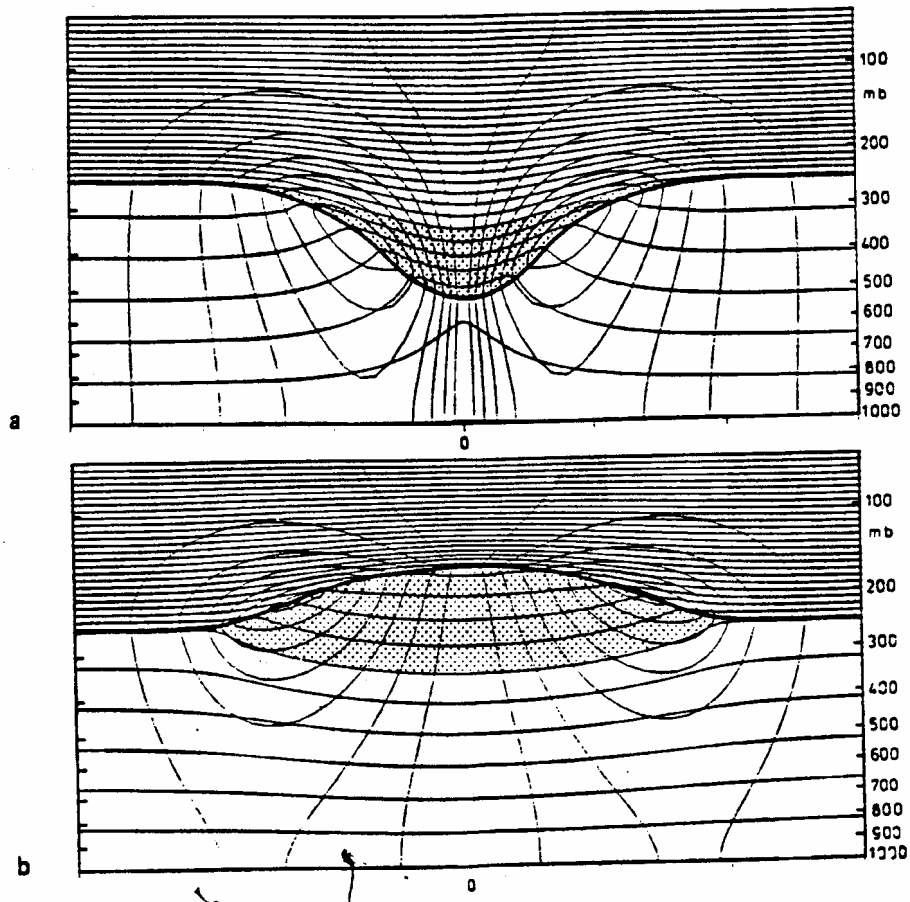


Figure PV1: (a) Positive *PV* anomaly (stippled). Potential temperature (quasi-horizontal lines), with the thick line representing the tropopause. Also shown are isotachs, with the + indicating flow into the page and —, flow out of the page. The horizontal axis spans 5000 km. (b) As in (a), except for a negative *PV* anomaly (from Hoskins et al. 1985).

Two case studies involving severe surface winds associated with extratropical cyclones are presented by Hoskins and Berrisford (1988) and Browning and Reynolds (1994). In each case, the investigators were able to identify a strong upper-level *PV* anomaly associated with a sharp lowering of the tropopause. Strong surface winds were noted to migrate in conjunction with the movement of this upper-level feature and were especially strong when the upper-level *PV* anomaly "locked onto" a region of low-level baroclinicity, as discussed above.

